Astro 404
Lecture 10
Sept. 15, 2021

Announcements:

- **Problem Set 3 due Friday**
  will need all of today's notes
- instructor office hours: today after class or by appt
- TA office hours: Thursday 2:30 - 3:30pm

Last time: stars as **self-gravitating spheres**
Q: what is enclosed mass $m(r)$*?* $m(0)$*?* $m(R)$*?*
Q: gravity field of a sphere at r*?*

mass density: $dm = \rho \; dV$

for sphere of radius R and total mass M

enclosed mass

$$m(r) = \int^{r} \rho \; dV = 4\pi \int_{0}^{r} \rho(r) \; r^2 \; dr$$

and thus for a spherical distribution

$$\rho(r) = \frac{1}{4\pi r^2} \frac{dm}{dr} \tag{1}$$

gravitational field at r:

$$\vec{g}(r) = -\frac{G \, m(r)}{r^2} \; \hat{r}$$

Gravitational Potential

gravitational potential Φ defined as

$$\vec{g} = -\nabla\Phi$$

which gives the potential in terms of the field as

$$\Phi = -\int \vec{g} \cdot d\vec{s}$$

up to a constant – usually set $\Phi \to 0$ as $r \to \infty$

for a spherically symmetric mass distribution the **gravitational potential energy** of a test mass m_{test} is

$$\Omega = m_{\text{test}}\Phi \overset{\text{sph}}{=} -\frac{G\ m(r)\ m_{\text{test}}}{r} \qquad (2)$$

ω *Q: significance of minus sign?*

Q: how to find gravitational potential energy of entire star?

Gravitational Potential Energy

for a spherically symmetric, continuous matter distribution
the **total gravitational potential energy** is (see Extras)
sum of contributions at each mass shell dm:

$$\Omega = -\int_0^M \frac{Gm}{r}\ dm = -G\int_0^R\ m(r)\ \rho(r)\ r\ dr \tag{3}$$

where the first integration is over the mass coordinate
Q: why the minus sign? physical significance?

result depends on stellar structure via $\rho(r)$ or $m(r)$
Q: order of magnitude for star of mass M and radius R?
Q: result for infinitely thin shell of size R?

Gravitational Potential Energy: Order of Magnitude

order of magnitude estimate from dimension analysis:
given M and R, and universal constant G
only one combination has units of energy: GM^2/R (check it!)

with the correct *minus sign to indicate a bound system*
we expect that

$$\Omega = -\alpha \frac{GM^2}{R} \tag{4}$$

where α is a dimensionless constant

example: an *infinitely thin shell* has

$$\Omega_{\text{shell}} = -\frac{GM^2}{R}$$

and we see that $\alpha_{\text{shell}} = 1$

Stellar Stability I

the Sun's size is constant on human timescales

$\Rightarrow$ not expanding, collapsing

$\Rightarrow$ stable

Why?

Appreciate: not a trivial result, could have been otherwise

compare with other gas blobs

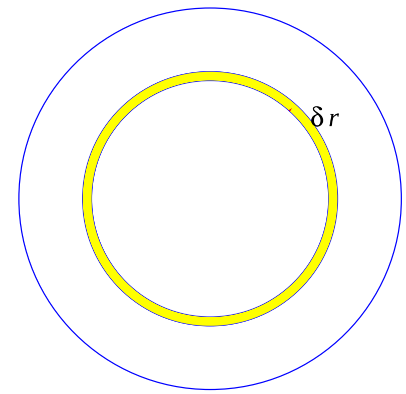
laboratory gases

- expand to fill available space

terrestrial and interstellar clouds

- irregular shapes
- morph with time

iClicker Poll: Forces on a Shell of Solar Gas



Consider a shell of gas in the Sun, **at rest**
i.e., Sun not expanding, contracting

How many forces are acting on this shell?

A zero

B only one

C more than one

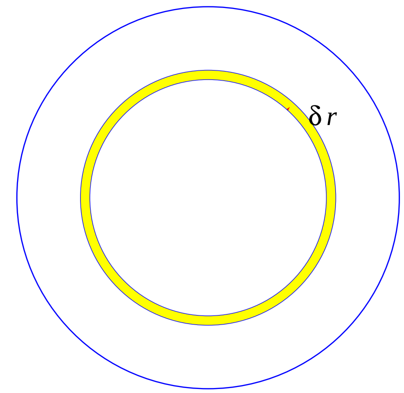
Consider a shell of gas in the Sun, **at rest**

radius r, thickness $\delta r \ll r$

shell area $A = 4\pi r^2$

shell volume

$$V = \frac{4\pi}{3}[(r+\delta r)^3 - r^3] \approx 4\pi r^2\,\delta r = A\,\delta r$$



shell mass $m_{\text{shell}} = \rho V = \rho A\ \delta r$

shell weight $F_{\text{w}} = -g m_{\text{shell}} = -g\rho A\ \delta r$:

downward force, but doesn't fall!?

Q: why? gas has weight–why not all at our feet?

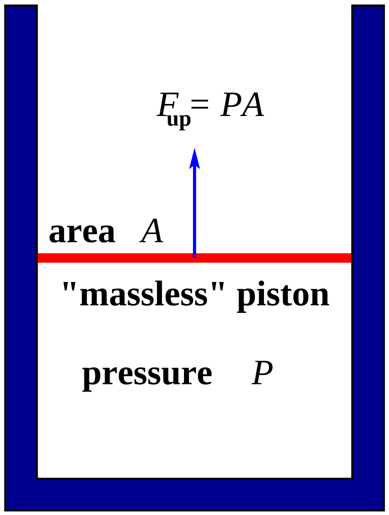
∞

Pressurized Stars

seemingly static nature of star sizes
means surface bulk is at rest
star surface has *zero acceleration* $\rightarrow$ *zero net force*
but gravity definitely present, so another force must exist!

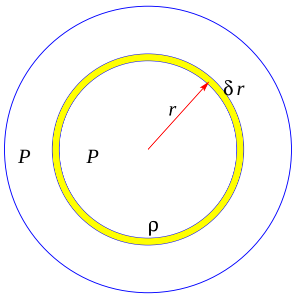
we know stars made of gas, at $T > 0$
so **gas pressure forces** certainly present
$\rightarrow$ a promising candidate to offset gravity

pressure is force per area: $P = F/A$
pressure force $F = PA$ is normal to area



iClicker Poll: Pressure in a Star

Consider a star with gas that everywhere has **constant, uniform pressure** P



for a shell of mass $\delta m = \rho \ A \ \delta r$ feeling gravity $g(r)$

What value of P will support this shell in a stable way?

A $P < \delta m \ g/A$

B $P = \delta m \ g/A$

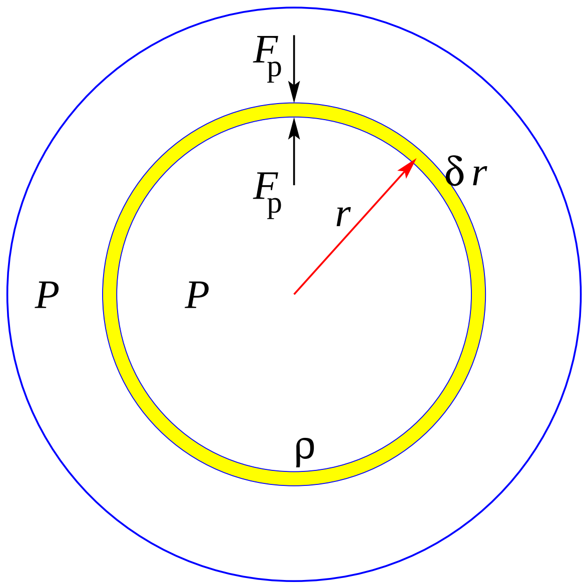
C $P > \delta m \ g/A$

D none of these will support the shell in a stable way

Uniform Pressure Star: Fails Uniformly!

consider a star with *uniform pressure P throughout*

for a *shell of area* A
pressure force from below: $F_{\text{up}} = PA$
this acts upward, oppose gravity. Yay!



but: above the shell,
pressure force: $F_{\text{down}} = PA$
same magnitude, opposite direction!

in this situation: $F_{\text{up}} - F_{\text{down}} = (P_{\text{up}} - P_{\text{down}})A = 0$
the pressure forces cancel!
pressure has no net effect!

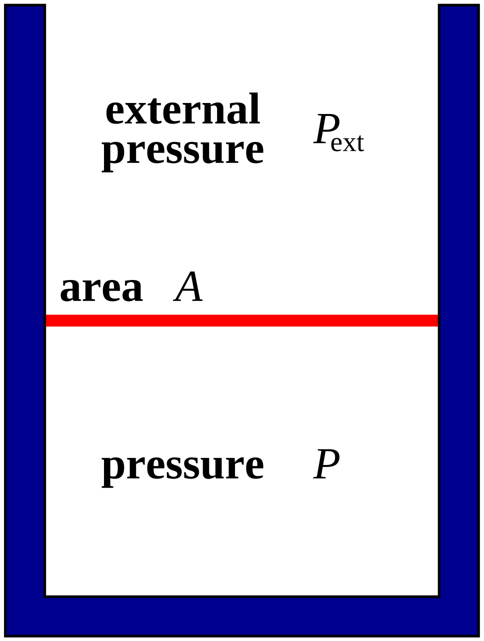
under uniform pressure, no net force!

in piston model:

if equal pressures inside and outside

$$\begin{aligned} P_{\text{ext}} &= P \\ F_{\text{down}} = P_{\text{ext}} A &= F_{\text{up}} = PA \\ F_{\text{net}} &= F_{\text{up}} - F_{\text{down}} = 0 \end{aligned}$$

no net force!



real life example: humans!

atmospheric pressure $P_{\text{atm}} \simeq 10^5$ Newton/m$^2 = 10^5$ Pa

- force on front of body $\approx 10^5$ Newton!
- would send you flying if unbalanced!
- but uniform pressure horizontally $\rightarrow$ forces cancel

Pressure Support in Real Life

yet we know in real life that
pressure can provide support, including against gravity!

- balloon: inward elastic force vs outward P
- car tire: pressure holds up car's entire weight!

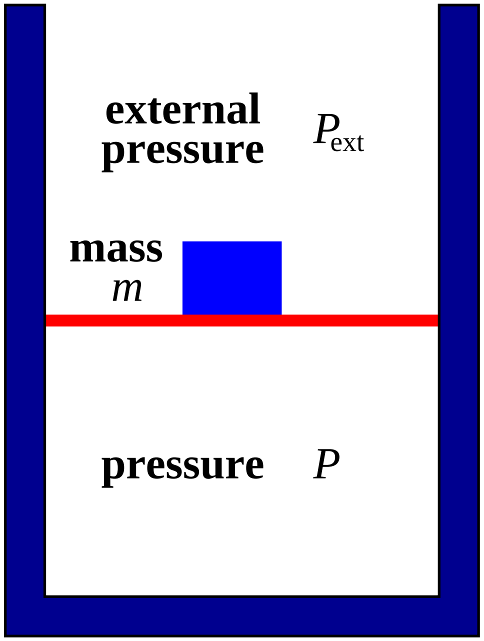
Q: where did we go wrong here? how to fix this?

Nonuniform Pressure: Net Force Emerges

a net pressure force *is* possible
if pressure is *nonuniform* inside the star
that is: $P(r) \neq const$

in piston model: add mass
weight exerts downward force

supported if
- **un**equal pressures inside and outside
- $P > P_{\text{ext}}$: pressure decreases upwards

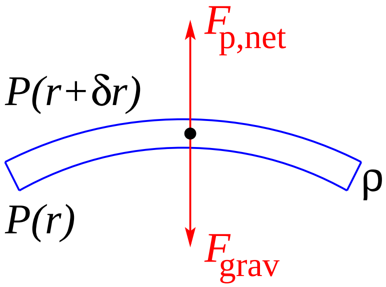


A Star With Nonuniform Pressure

pressure forces on gas shell at $(r, r + \delta r)$

- from below, upward pressure $P_{\text{up}} = P(r)$
- from above $P_{\text{down}} = P(r + \delta r)$

net upward force is



$$F_{\text{net,up}} = (P_{\text{up}} - P_{\text{down}})\ A$$

$$= [P(r) - P(r + \delta r)]\ A \qquad (5)$$

$$= -\frac{dP}{dr}\ \delta r\ A \qquad (6)$$

nonzero if pressure not constant!

Q: why the sign? what does this mean physically?

Q: mathematical condition for a stationary star?

Hydrostatic Equilibrium

net pressure force on shell:

$$F_{\text{net,up}} = -dP/dr \; \delta r \; A \tag{7}$$

upward direction if $dP/dr < 0$: $\rightarrow$ pressure decreases outward, increases inward so on Earth, air is thin at altitude!

net gravity force = *weight* of shell:

$$F_{\text{weight}} = \delta m \; g(r) = \rho(r) \; g(r) \; A\delta r \tag{8}$$

when forces balance: star attains hydrostatic equilibrium: pressure gradient exactly balances downward gravity for every mass shell in star:

$$F_{\text{pressure}} = F_{\text{weight}} \tag{9}$$

$$-\frac{dP}{dr} \; A \; \delta r = \rho(r) \; g(r) \; A \; \delta r \tag{10}$$

shell volume cancels!

The Mighty Equation of Hydrostatic Equilibrium

for a spherical star in hydrostatic equilibrium

$$\frac{dP}{dr} = -g\rho = -\frac{G\ m(r)\ \rho(r)}{r^2}$$

Lessons:

- given density $\rho(r)$, and hence $m(r)$: this determines pressure
- a star's mechanical structure $\rho(r), m(r)$ intimately related to thermal structure via pressure profile $P(r)$

Q: how to solve for $P(r)$? boundary conditions?

Stellar Pressure

hydrostatic equilibrium: pressure gradient balances gravity

$$\frac{dP}{dr} = -\frac{G\ m(r)\ \rho(r)}{r^2} \tag{11}$$

integrate to solve for pressure

$$\begin{aligned} \int_0^r \frac{dP}{dr} dr &= P(r) - P(0) && (12) \\ &= -\int_0^r \frac{G\ m(r)\ \rho(r)}{r^2} dr && (13) \\ & && (14) \end{aligned}$$

integration requires *boundary conditions*

- $P(0) = P_c$ pressure at center: **central pressure**
- $P(R)$ pressure at surface

iClicker Poll: Surface Pressure

Consider a star, mass M and radius R, in hydrostatic equilibrium

What what is pressure $P(R)$ at surface boundary?

A $P(R) < 0$

B $P(R) = 0$

C $P(R) > 0$

D none of the above

Pressure at the Extremes

if star fully in hydrostatic equilibrium
pressure gradient balances gravity: $dP/dr = -Gm\rho/r^2$

- outer boundary defined by $\rho(R) = 0$
- so there, $dP/dr = 0$: pressure minimized $\rightarrow$ $P(R) = 0$

thus the integration

$$P(R) - P(0) = -\int_0^R \frac{G\ m(r)\ \rho(r)}{r^2} dr \tag{15}$$

gives the *central pressure*

$$P_c = P(0) = \int_0^R \frac{G\ m(r)\ \rho(r)}{r^2} dr \tag{16}$$

and thus

$$P(r) = P_c - \int_0^r \frac{G\ m(r)\ \rho(r)}{r^2} dr \tag{17}$$

pressure drops monotonically from the central value

Hydrostatic Equilibrium: Mass Coordinate Picture

recall we can label star interior via radius
but also by *mass coordinate* $dm = 4\pi r^2\ \rho\ dr$
where $m \in [0, M]$, and where $r(m)$ varies for different density profiles ρ

in these coordinates:

$$\frac{dP}{dr} = \frac{dP}{dm}\ \frac{dm}{dr} = 4\pi r^2 \frac{dP}{dm} \tag{18}$$

and so hydrostatic equilibrium $dP/dr = -G\rho m/r^2$ gives

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \tag{19}$$

$$P = P_c - \int_0^M \frac{Gm\ dm}{4\pi r^4} \tag{20}$$

Q: order of magnitude of central pressure P_c?

Central Pressure: Order of Magnitude

order of magnitude:

- find *scaling* with parameters M, R, and constants like G
- ignore dimensionless constants like 2, π, etc

estimate *central pressure:*

$$P_c \sim \frac{\text{characteristic force}}{\text{characteristic area}} \tag{21}$$

$$\sim \frac{M\, g(R)}{R^2} = \frac{GM^2/R^2}{R^2} = \frac{GM^2}{R^4} \tag{22}$$

or try characteristic energy GM^2/R per volume R^3: same answer!

or estimate from integral: same answer

$$P_c = \frac{1}{4\pi}\int_0^M \frac{Gm\ dm}{r^4} \sim \frac{GM^2}{R^4} \tag{23}$$

Q: is the answer physically reasonable–scaling with M, R?

Limits to Pressure

central pressure exact expression

$$P_{\mathrm{c}} = \frac{1}{4\pi} \int_0^M \frac{Gm\ dm}{r^4} \tag{24}$$

note denominator $r^4 < R^4$; use this to set a *strict lower bound*

$$P_{\mathrm{c}} > P_{\mathrm{c,min}} = \frac{1}{4\pi} \int_0^M \frac{Gm\ dm}{R^4} = \frac{GM^2}{8\pi R^4} \tag{25}$$

order of magnitude estimate with $1/8\pi$ factor

Q: for which stars is P_c smallest? largest?

Central Pressure of Stars

plug in numbers for mass and radius:

$$P_{\mathrm{c,min}} \stackrel{\text{Sun}}{=} 5 \times 10^{13}\ \mathrm{N/m^2} = 4 \times 10^{8}\ \mathrm{atm}$$

$$\stackrel{\text{Sirius B}}{=} 9 \times 10^{21}\ \mathrm{N/m^2} = 10^{17}\ \mathrm{atm}\ \text{white dwarf}$$

- solar central pressure huge!
- white dwarfs have similar mass but much smaller
  billions of times larger still!
- giants and supergiants: limit much smaller

Ideal Gases

ideal gas: free particles in thermal equilibrium
particles in constant random motion in all directions

ideal gas pressure P, temperature T, and density ρ
not all independent: related by **ideal gas equation of state**

$$PV = N\,kT \tag{26}$$

for a fluid element with volume V and number N of particles
note: in physics and astronomy, N counts particles one by one
but chemists count in units of moles, which gives $PV = \mathcal{N}_{\rm mole}\mathcal{R}T$

k: Boltzmann's constant

Ideal Gas Equation of State

starting with the familiar version

$$PV = N\,kT \tag{27}$$

introduce the **number density** of gas particles

$$n = \frac{N}{V} \tag{28}$$

the number of particles per unit volume

thus we can write

$$P = n\,kT \tag{29}$$

Q: how is this related to the mass density ρ *of the gas?*

Ideal Gases: Mass and Energy Densities

for gas fluid element of mass M, with particle number N
mass density

$$\rho = \frac{M}{V} = \frac{M}{N}\frac{N}{V} = m_{\rm g}\ n \tag{30}$$

where $m_{\rm g}$ is *average mass of one gas particle*

so ideal gas equation of state is

$$P = n\ kT = \frac{\rho\, kT}{m_{\rm g}} \tag{31}$$

- pressure depends on **both** density and temperature: $P \propto \rho T$
- given any two of (ρ, P, T), gas law gives the third

Director's Cut Extras

Potential Energy of a Spherical Mass Distribution

For spherical continuous (i.e., with density ρ) mass distribution

$$\frac{d\Phi}{dr} = -\frac{Gm(r)}{r^2} \tag{32}$$

from this we want to find the gravitational potential energy

For a set of point masses:
the potential energy is the sum over distinct pairs

$$\Omega = -\frac{1}{2}\sum_i \sum_{j \neq i} \frac{Gm_i m_j}{r_{ij}} \tag{33}$$

where

- the double sum is over $i, j = 1, \ldots, N$ particles and omits identical pairs $j = i$
- $r_{ij} = |\vec{r}_i - \vec{r}_j|$ is the distance between i and j
- and the factor of 1/2 corrects for double counting of pairs

generalizing to a smooth distribution of mass, we have

$$\Omega = -\frac{1}{2}\int \rho \ \Phi \ dV \tag{34}$$

for our spherically symmetric case
we can use the mass coordinate $dm = \rho \ dV$:

$$\Omega = -\frac{1}{2}\int \Phi \ dm \tag{35}$$

and then we integrate by parts

$$\Omega = \frac{1}{2}\int m \ d\Phi = \frac{1}{2}\int m \ \frac{d\Phi}{dr}dr \tag{36}$$

and now using the relation above

$$\Omega = -\frac{1}{2}\int \frac{Gm^2}{r^2}dr \tag{37}$$

we have

$$\Omega = -\frac{1}{2}\int \frac{Gm^2}{r^2}dr \qquad (38)$$

integrating by parts again, we have: $\int u\ dv = uv - \int v\ du$
here $u = Gm^2/2$, $dv = -dr/r^2 = d(1/r)$
and the uv term is $Gm^2/2r|_0^\infty = 0$

so we finally have

$$\Omega = -\int \frac{Gm}{r}dm \qquad (39)$$

which was to be shewn,
and where the prefactor of 1/2 is canceled by a factor of 2 from the differential of m^2