

Astro 404
Lecture 11
Sept. 17, 2021

Announcements:

- **Problem Set 3 due online 5pm today**
- **Problem Set 4 due next Friday**

Last time:

- hydrostatic equilibrium

Q: why hydro? why static? why equilibrium?

Q: what's the mighty equation?

Q: what would the Sun do if gravity switched off ($G = 0$)?

- ideal gas equation of state

Q: mighty equation? in terms of mass density ρ ?

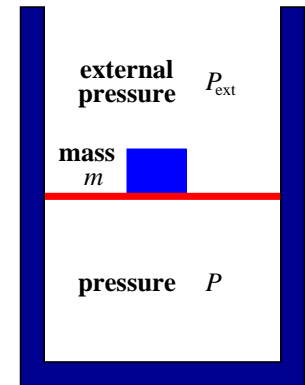
hydrostatic equilibrium:

hydro: applies to (compressible) fluid = gas

static: applies to a system at rest (no bulk motion)

equilibrium: forces are exactly balanced

downward gravity vs upward pressure gradient



the mighty equation

$$\frac{dP}{dr} = -g\rho = -\frac{G m(r) \rho(r)}{r^2}$$

ideal gas equation of state

$$PV = NkT \quad (1)$$

$$P = n kT \quad (2)$$

$$= \frac{\rho kT}{m_g} \quad (3)$$

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with number density $n = N/V$, average gas particle mass m_g

Ideal Gas: Macro vs Micro Pictures

gases on microscopic scale

a swarm of particles, for example atoms or molecules

- gas particles have *empty space between them*
not packed together as in liquid or solid
- gas particles are in *constant random motion*
act as free particles (constant velocity) between collisions
- *collide elastically with each other, container walls* (if any)
exchange energy & momentum → distribution of speeds

ω on macroscopic scales (i.e., how we see things)
particle motions perceived as *temperature*

Poll: Gas Particle Speeds

consider a parcel of gas:

- macroscopically, gas is at rest (not moving/blowing)
- at room temperature T

in this gas:

the average particle **velocity** \vec{v} and **speed** $v = |\vec{v}|$ are:

A $\vec{v} = 0$ and $v = 0$

B $\vec{v} = 0$ and $v > 0$

C $\vec{v} \neq 0$ and $v = 0$

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D $\vec{v} \neq 0$ and $v > 0$

Gasses: Average Properties

average particle **velocity vector** vanishes: $\langle \vec{v} \rangle = 0$

why? *not* because particles are still

rather: equal numbers with $v_x > 0$ vs $v_x < 0 \rightarrow$ averages to zero

otherwise: gas would have net v_x , wouldn't be at rest!

note **microscopic–macroscopic (particle–bulk) correspondence:**

micro: equal probabilities for particle $\vec{v} > 0$ and $\vec{v} < 0$

that is: no preferred direction

macro: corresponds to bulk gas speed $\vec{u}_{\text{gas}} = 0$

since particles are moving, speeds $\langle v^2 \rangle = \langle v_x^2 \rangle + \langle v_y^2 \rangle + \langle v_z^2 \rangle > 0$

\rightarrow average kinetic energy of each gas particle is nonzero

Ideal Gas: Microscopic Picture

consider ideal gas of particles with mass m_g
in cubic box of size L and volume $V = L^3$

www: ideal gas simulation

a particle of speed v_x bounces between yz walls

- bounce return time for same wall: $\delta t = 2L/v_x$
- bounces are *elastic*: $v_x^{\text{after}} = -v_x^{\text{before}}$
so $\delta v_x = v_x^{\text{after}} - v_x^{\text{before}} = 2v_x$, and $\delta p_x = 2m_g v_x$

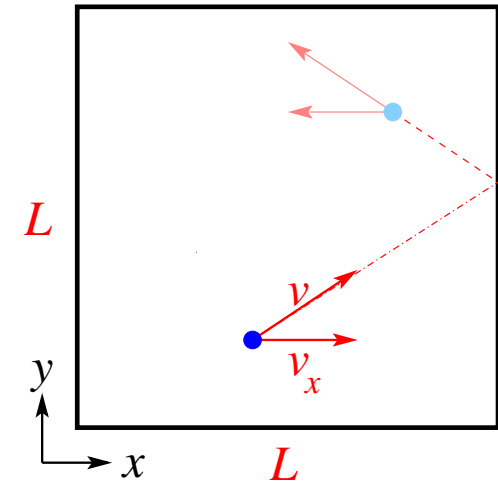
so collision changes particle momentum!

this exerts *force on wall*

$$F_x = \frac{\delta p_x}{\delta t} = \frac{2m_g v_x}{2L/v_x} = \frac{m_g v_x^2}{L} \quad (4)$$

Q: force per unit area in single particle collision?

Q: total force per area due to N particles?



for particle of mass m_g and speed v_x :

force per unit area of wall

$$\frac{F_x}{A} = \frac{F_x}{L^2} = \frac{m_g v_x^2}{L^3} = \frac{m_g v_x^2}{V} \quad (5)$$

total force per area: *pressure!*

sums over all N particles

$$P = \sum_{i=1}^N \frac{m_g v_{i,x}^2}{V} = N \frac{m_g \langle v_x^2 \rangle}{V} \quad (6)$$

where $\langle v_x^2 \rangle$ means *average v_x^2 over all particles*

gas speed distribution is *isotropic* = same in all directions

Q: *what does this means for $\langle v_x^2 \rangle$ vs $\langle v_y^2 \rangle$? for $\langle v_{\text{tot}}^2 \rangle$?*

Ideal Gas: Equation of State Emerges!

if gas speeds are *isotropic = no preferred direction*, then

- $\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle$
- $\langle v^2 \rangle = \langle v_x^2 + v_y^2 + v_z^2 \rangle = 3\langle v_x^2 \rangle$

and so the pressure is

$$P = N \frac{m g \langle v_x^2 \rangle}{V} = N \frac{m g \langle v^2 \rangle / 3}{V} \quad (7)$$

woo hoo! our microscopic collision theory shows

- $P \propto 1/V$ **check!** Q: *microscopic physical reason?*
- $P \propto N$ **check!** Q: *microscopic physical reason?*

∞ to fully match ideal gas equation of state
what does this mean about temperature?

microscopic theory gives ideal gas pressure

$$P = N \frac{m_g \langle v^2 \rangle / 3}{V} \quad (8)$$

- $P \propto N$: more particles \rightarrow collisions more frequent
- $P \propto 1/V$: more $V \rightarrow$ collisions less frequent

and so to match ideal gas law $P = N kT / V$ we find

$$kT = \frac{1}{3} m_g \langle v^2 \rangle \quad (9)$$

temperature proportional to square of average particle speed

typical gas particle speed (root-mean-square):

$$v_{\text{rms}} = \sqrt{\frac{3kT}{m_g}} \quad (10)$$

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hotter gas \leftrightarrow faster particles

Ideal Gas Internal Energy

internal energy sum all gas particle

- kinetic energy, plus
- rotational energy if any

for a *monatomic* ideal gas (non-rotating particles)
with N particles at temperature T
non-relativistic internal energy is

$$U = \frac{1}{2}Nm\langle v^2 \rangle = \frac{3}{2}N kT \quad (11)$$

ideal internal energy only depends on temperature T
internal energy also known as "*thermal energy*"

Q: what is energy per unit volume?

Ideal Gas Energy Density

ideal gas internal energy

$$U = \frac{3}{2}N kT = \frac{3}{2}PV \quad (12)$$

so ideal gas *inter energy per unit volume*
or **internal energy density**, for monatomic gas

$$\varepsilon = \frac{U}{V} = \frac{3}{2}n kT = \frac{3}{2}P \quad (13)$$

will be useful to define internal energy per unit mass

$$u = \frac{U}{Nm_g} = \frac{\varepsilon}{\rho} = \frac{3kT}{2m_g} \quad (14)$$

Stars Average Temperature

a star's *total internal energy*

$$U = \int \varepsilon dV = \int \frac{\varepsilon}{\rho} \rho dV = \int u dm = \langle u \rangle M \quad (15)$$

where $\langle u \rangle$ is average internal energy per unit mass

but for ideal gas, $u = 3/2 P/\rho = 3kT/2m_g$, so

$$U = \frac{3 M}{2 m_g} \langle kT \rangle \quad (16)$$

and thus **average temperature** is

$$\langle kT \rangle = \frac{2 m_g U}{3 M} \quad (17)$$

Stellar Energy Budget: Equilibrium

hydrostatic equilibrium condition

$$dP = -\frac{G m(r)}{r^2} \rho(r) dr \quad (18)$$

multiply by $V = 4\pi/3 r^3$ and integrate
left side:

$$\int_0^R V dP = [PV]_0^R - \int_0^R P dV \quad (19)$$

but $[PV]_0^R = 0$: why?, so we have

$$\int_0^R V dP = - \int_0^R P dV = -\frac{2}{3} \int_0^R \epsilon dV = -\frac{2}{3} \int_0^R \epsilon dV = -\frac{2}{3} U \quad (20)$$

righthand side:

$$-\frac{1}{3} \int \frac{Gm}{r} \rho 4\pi r^2 dr = -\frac{1}{3} \int \frac{Gm dm}{r} = \frac{1}{3} \Omega \quad (21)$$

The Virial Theorem

for an ideal gas in hydrostatic equilibrium:

$$U = -\frac{1}{2}\Omega \quad (22)$$

internal energy is minus half gravitational potential energy:

the Virial theorem

this is a powerful tool for understanding stellar equilibria

Virial application I: average stellar temperature

$$\langle kT \rangle = \frac{2 m_g U}{3 M} = \frac{1 m_g \Omega}{3 M} \quad (23)$$

$$= \frac{1 m_g}{3 M} \int \frac{Gm dm}{r} \quad (24)$$

Q: order of magnitude for star of mass M , radius R ?

Average Temperature

to order of magnitude,

$$\langle kT \rangle = \frac{1}{3} \frac{m_g}{M} \int \frac{Gm \, dm}{r} \sim \frac{GMm_g}{R} \quad (25)$$

Q: *check—reasonable dependence on M? on R?*

plug in numbers for the Sun:

$$\langle kT_{\odot} \rangle \sim 10^3 \text{ eV} = 1 \text{ keV} \quad (26)$$

$$\langle T_{\odot} \rangle \sim 10^7 \text{ K} = 10 \text{ MK} \quad (27)$$

where eV = electron volt = $e \cdot 1 \text{ Volt} = 1.602 \times 10^{-19} \text{ Joule}$

Q: *comparison with surface temperature?*

Q: *implications?*

Lessons: Stellar Interior vs Surface Temperature

for the Sun: $\langle T_{\odot} \rangle \sim 10 \text{ MK}$

compare to surface effective $T_{\text{eff}} \approx 5800 \text{ K}$

- *surface T_{eff} not representative of stellar average!*
- *stellar interiors much hotter than surface!*

in energy units: temperature $kT \gg 1 \text{ eV}$ atomic binding energies
atoms are unbound inside of stars

→ *most of stellar interiors are ionized plasma!*

Virial Theorem Application II: Total Energy

Virial theorem relates

- **gravitational potential energy** Ω , and
- **internal energy** U

thus the **total energy** in a star under hydrostatic equilibrium is

$$E_{\text{tot}} = U + \Omega = \frac{1}{2}\Omega = -\frac{1}{2} \int \frac{Gm \, dm}{r} \quad (28)$$

Q: sign of Ω ? sign of E_{tot} ? significance?

Q: what if system loses energy (spoiler: radiates) ?

Implications of the Virial Theorem

Virial theorem: in equilibrium

gravitational potential energy and internal energy are *related*

$$E_{\text{tot}} = U + \Omega = \frac{1}{2}\Omega \quad (29)$$

- $\Omega = -\int Gm/r, dm/r < 0$: *gravitational binding*
- $U = -\Omega/2$: *more bound \rightarrow more internal energy*
- $E_{\text{tot}} = \Omega/2 = -U < 0$: *system is gravitationally bound*
i.e., must supply energy to unbind and move to $r \rightarrow \infty$

if system loses energy while keeping in equilibrium

E_{tot} gets smaller \rightarrow more negative, and so

- $|\Omega|$ larger Q: *and so?*
- U larger \rightarrow more internal energy Q: *why is this bizarre?*

a star in hydrostatic equilibrium that gradually *loses energy*:

- must increase $|\Omega| \sim GM^2/R$
→ must become *more compact!*
- also must increase internal energy U
but $\langle T \rangle \propto U$: star must get *hotter!*

stars get hotter when they lose energy!?!

stars have “negative specific heat”

but not so strange:

if stars become more compact, gravity stronger

so must heat to increase pressure and maintain hydrostatic equilb.

Ideal Gas: Ultra-Relativistic Case

thus far implicitly assumed *gas speeds are non-relativistic*

- speeds $v \ll c$ and $p \ll m_g c$
- so $p^2/m_g \ll m_g c^2$ and thus $kT \ll m_g c^2$

now consider opposite limit: **ultra-relativistic** particles

- $v \approx c$ or even $v = c$ Q: examples?
- relativistic momentum $p \gg m_g c$
- energy $E = \sqrt{(cp)^2 + (m_g c^2)^2} \approx cp$

revisit microscopic pressure derivation

$$P_{ur} = \frac{N \langle pv \rangle}{3V} \approx \frac{1}{3} \frac{N \langle E \rangle}{V} = \frac{1}{3} \frac{U_{ur}}{V} \quad (30)$$

and thus for a relativistic gas $PV = U/3$, and energy density is

$$\epsilon_{ur} = 3P_{ur} \quad (31)$$