Astro 404

Lecture 12

Sept. 20, 2021

Announcements:

- **Problem Set 4 due Friday**

  instructor office hours: Wed 11am-noon or by appt

  TA office hours: Thurs 2:30-3:30

Last time:

- Ideal gas theory

Q: microscopic origin of pressure? temperature?

- Virial theorem

relates total internal energy U

and gravitational potential energy Ω

Q: namely? Under what conditions?

Ideal Gas Recap

microscopic theory gives ideal gas pressure

$$P = \frac{N}{V}\frac{m_{\text{g}}\langle v^2\rangle}{3} = \frac{N}{V}kT = n\ kT \qquad (1)$$

- $P \propto N$: more particles $\leftrightarrow$ collisions more frequent
- $P \propto 1/V$: more V $\leftrightarrow$ collisions less frequent
- $kT = m_{\text{g}}\langle v^2\rangle/3$: faster particles $\leftrightarrow$ hotter gas

Virial Theorem Recap

the Virial theorem relates global stellar energy reservoirs

$$U = -\frac{1}{2}\Omega \tag{2}$$

total internal energy is minus half total grav potential energy

applies to: *monatomic ideal gas* in *hydrostatic equilibrium*

a kind of *stellar equipartition*

like ideal gas kinetic equipartition: $m_{\rm g}\langle v_x\rangle^2/2 = m_{\rm g}\langle v_y\rangle^2/2 = m_{\rm g}\langle v_z\rangle^2/2 = kT$

the **total energy** in a star under hydrostatic equilibrium is

$$E_{\rm tot} = U + \Omega = \frac{1}{2}\Omega = -\frac{1}{2}\int \frac{Gm\ dm}{r} \tag{3}$$

Q: order of magnitude for each?

Virial Theorem: Order of Magnitude

to order of magnitude:

- internal/thermal energy: $U \sim M\langle kT \rangle / m_{\rm g} = N\langle kT \rangle$
- gravitational potential energy: $\Omega \sim -GM^2/R$

Virial theorem links these, and implies

$$kT \sim \frac{GMm_{\rm g}}{R} \tag{4}$$

note: $kT \sim m_{\rm g}\langle v^2 \rangle$, and $GM/R \sim \Phi$
so Virial also means that

$$\langle v^2 \rangle \sim \Phi \tag{5}$$

⇒ particle thermal speeds set by depth of gravitational potential

Virial Theorem and Stellar Evolution

note that Virial theorem relates

- *global stellar energy reservoirs*: gravitational and internal/thermal
- and hence total energy localized to the star
- while in state of *hydrostatic equilibrium*

but recall that stars lose energy – they are luminous!

Q: how can a star do this and maintain equilibrium state?

Maintaining Equilibrium: Burning Phases

Virial theorem: $\langle kT \rangle \sim GMm_{\rm g}/R$, and
$E_{\rm tot} \sim -N \langle kT \rangle \sim -GM^2/R$

to maintain $E_{\rm tot}$ → must maintain R and T

- the star must *tap a non-gravitational, non-thermal "fuel source"* that supplies energy to maintain T and power luminosity
- a given equilibrium state (R, T, L) lasts as long as fuel permits

Thus we anticipate **burning phases set by fuel supplies**

Q: when will *a star change its equilibrium state?*

Seeking Equilibrium: Contraction Phases

if the "fuel source" is gravitational energy itself
radiated energy *decreases* $E_{\rm tot}$

But $E_{\rm tot} < 0$ already: star is gravitationally bound
- radiative energy loss makes $E_{\rm tot}$ *more negative*
- and thus $|E_{\rm tot}|$ *increases*

Virial theorem $|E_{\rm tot}| \sim GM^2/R \sim NkT/m_{\rm g}$
demands that R decreases as well
- *the star contracts!*
- and in response *heats up!*

this continues unless/until a new fuel source ignites

Thus we expect **contraction phases** between burning phases

Perturbing a Star

consider a star maintaining hydrostatic equilibrium
by burning a non-gravitational, non-thermal fuel supply
to replenish thermal energy balancing losses through lumonosity

now imagine a *perturbation that generates energy faster*
due to upward density or temperature fluctuation
at the star's center ("core")

if the star is composed of an *ideal gas*:
Q: how does the star's center respond?
Q: what is the effect on the fuel burning rate?
Q: what is the net response to the perturbation?

∞

Q: what if a perturbation had lowered the core burning rate?

The Virial Theorem and Stellar Stability

an *increased core energy production rate*:

▷ leads to a higher core temperature
▷ for ideal gas: increases pressure
▷ pressure gradient drives *core expansion*
▷ density drops, pressure and temperature drop
▷ this *lowers burning rate* until
▷ *the star recovers its initial state!*

can convince yourself:
same conclusion for downward perturbation

Lesson: perturbed ideal gas stars burning non-gravitational fuel
are driven back to initial state
the equilibrium is stable!

The Virial Theorem and Stellar Contraction

argument courtesy of Alessandro Chieffi

consider a star not generating energy via "fuel burning"
and thus contracting

if contraction isn't too fast
star passes through a series of states near hydrostatic equilibrium
in which total energy is

$$E_{\text{tot}} = -U_{\text{int}} \sim -N \langle kT \rangle \tag{6}$$

transition between states requires change of internal energy
this *takes time to occur*

"protects" the star against sudden violent changes

Poll: Ultra-Relativistic Stars

so far, implicitly assumed stars are *non-relativistic* ideal gasses

Consider a star composed of a gas of *relativistic particles*
that is: speeds $v \approx c$ or even $v = c$

Vote your conscience!
Will the star be more or less stable than if non-relativistic?

A more stable (faster particles = more internal energy = more tightly bound)

B less stable (faster = more pressure = less tightly bound)

C no change in stability (effects cancel)

Ideal Gas: Ultra-Relativistic Case

thus far implicitly assumed *gas speeds are non-relativistic*

- speeds $v \ll c$ and $p \ll m_{\rm g} c$
- so $p^2/m_{\rm g} \ll m_{\rm g} c^2$ and thus $kT \ll m_{\rm g} c^2$

now consider opposite limit: **ultra-relativistic** particles

- $v \approx c$ or even $v = c$ *Q: examples?*
- relativistic momentum $p \gg m_{\rm g} c$
- energy $E = \sqrt{(cp)^2 + (m_{\rm g} c^2)^2} \approx cp$

revisit microscopic pressure derivation

$$P_{\rm rel} = \frac{N \langle pv \rangle_{\rm rel}}{3V} \approx \frac{1}{3} \frac{N \langle E \rangle}{V} = \frac{1}{3} \frac{U_{\rm rel}}{V} \tag{7}$$

and thus for a relativistic gas $PV = U/3$, and *energy density* is

$$\varepsilon_{\rm rel} = 3 P_{\rm rel} \tag{8}$$

Relativistic Gas Example: Blackbody Photons

photons are massless, have $v_\gamma = c$: always relativistic!
a "gas" photons in equilibrium at T: **blackbody radiation**
$\Rightarrow$ most important example of relativisitc gas

blackbody properties depend only on T:

$$\begin{aligned} \text{flux} \quad & F = \sigma_{\mathrm{SB}} T^4 \\ \text{energy density} \quad & \varepsilon = a_{\mathrm{SB}} T^4 \\ \text{pressure} \quad & P = \tfrac{1}{3} a_{\mathrm{SB}} T^4 = \varepsilon/3 \end{aligned} \tag{9}$$

where the two Stefan-Botlzmann constants are

- $\sigma_{\mathrm{SB}} = \pi^2 k^4 / 60 \hbar^3 c^2 = 5.67 \times 10^{-5}\ \mathrm{erg\,cm^{-2}\,s^{-1}\,K^{-4}}$
- $a_{\mathrm{SB}} = 4\sigma/c = \pi^2 k^4 / 15 \hbar^3 c^3 = 7.57 \times 10^{-15}\ \mathrm{erg\ cm^{-3}\ K^{-4}}$

we see that indeed photon energy density and pressure are indeed related by $\varepsilon_\gamma = 3P_\gamma$

Virial Theorem for an Ultra-Relativistic Gas

basic Virial theorem argument still holds

$$\int P \, dV = -\frac{1}{3}\Omega_{\rm grav} \tag{10}$$

but for **relativistic gas:** pressure and energy density related via $P_{\rm rel} = \varepsilon_{\rm rel}/3$

instead of non-relativistic $P_{\rm nr} = (2/3)\varepsilon_{\rm nr}$

so $\int P_{\rm rel} \, dV = (1/3) \int \varepsilon_{\rm rel} \, dV = U_{\rm rel}/3$

thus for a relativistic gas, Virial theorem is

$$U_{\rm rel} = -\Omega_{\rm grav} \tag{11}$$

Q: so what is total energy? implications?

Q: for a fixed mass distribution = fixed Ω, which gas is hotter?

Ultra-Relativistic Stars are Unstable

relativistic gas in equilibrium: Virial theorem

$$U_{\text{rel}} = -\Omega_{\text{grav}} \tag{12}$$

this gives total energy

$$E_{\text{tot}} = \text{internal} + \text{grav pot} = U_{\text{rel}} + \Omega = 0 \tag{13}$$

total energy is zero!

dramatic implications:

- $E_{\text{tot}} = 0$ means system is marginally bound
- transition between equilibrium states requires no change in internal energy
- star can evolve violently: **a relativistic star is unstable!**

Virial Theorem: Lessons

equilibrium links thermal and gravitational energy
more compact $\leftrightarrow$ hotter

- stellar interiors much hotter than $T_{\rm eff}$
- as stars lose energy they get hotter!
- (non-relativistic) ideal gas stars are self-regulating: stable
- (non-relativistic) ideal gas stars require time to evolve
- relativistic stars are barely bound, can evolve rapidly
  these stars are unstable!

Stars: Energy Generation

How Does the Sun Shine?

The Sun radiates: shines from thermal radiation

- recall: surface flux $F_{\text{surf},\odot} = \sigma T^4_{\text{surf},\odot} = 60\ \text{MWatt/m}^2$

- total power output = rate of energy emission = luminosity

$$L_\odot = 4\pi R^2_{1\ \text{AU}} F_\odot(1\ \text{AU}) = 3.85 \times 10^{26}\ \text{Watts} \tag{14}$$

$\rightarrow$ the Sun is a 4×10^{26}-Watt lightbulb

- But also: the Sun has *constant* temperature, luminosity (over human timescales $\gtrsim$ centuries)

Q: how is the Sun unlike a cup of coffee?

The Sun is Not a Cup of Coffee

Coffee Thermodynamics

Demo: cup of coffee: cools

thermodynamic lesson:

- left alone, hot coffee cools (surprise!)
  $\rightarrow$ energy radiated, not replaced
- to keep your double-shot soy latte from cooling
  need Mr. Coffee™ machine–energy (heat) source

Contrast with the Sun

- surface $T_\odot$ constant over human lifetimes
  but energy *is* radiated, at enormous rate
- ergo: something must replace the lost energy

▷ What is solar heat source (fuel supply)?
$\rightarrow$ a mystery in Astronomy until the 20th century

Q: all possible energy/heat sources which Sun taps?

Q: how to test/compare which are important?

Energy Conservation and the Sun

recall: power is energy flow rate $L = dE/dt$

assume:

- Sun always emits energy at today's rate (L constant)
- radiation lasts for time $\tau_\odot$ = "lifetime" of Sun

  Q: what is a minimum value for $\tau_\odot$*?*

energy output over Sun's lifetime:

$$E_{\text{lost}} = L\tau$$

Energy conservation:

solar energy supply = lifelong energy output

Solar Batteries: Required Lifetime

we found from radioactive dating of meteorites:
the solar system is very old: age $t_{\rm ss} = 4.55 \times 10^9 \ yr$
Sun's present age essentially the same:
$t_{\odot,\rm now} = t_{\rm ss} = 4.55$ billion years

total energy output over this time is huuuge!
$\rightarrow$ required huge energy reservoir

in other words: important solar energy source(s) $\equiv$ long-lived:
$\tau_{\rm source} = E_{\rm res}/L_\odot = \tau_\odot > t_{\odot,\rm now} \approx 5$ billion yr

Q: possible sources–not just right answer, but any energy reservoirs?