Astro 404

Lecture 20

Oct. 8, 2021

Announcements:

- Good news: *no homework due next Friday!*
- *Bad news:* **Hour Exam Friday Oct 18.** Info on Canvas
  all homework solutions are posted or will be
- Overview today, Review on Wednesday
  but new material today and this week is not only exam
  and Instructor will answer Canvas Exam Discussion Forum

Midterm Exam Format

Exam duration: 50 minutes
During class time: 10:00-10:50am, Friday Oct 15

Formats
- Traditional in-person: Loomis 144. TA will proctor
- Online: download questions, upload scanned answers
  Instructor will proctor
  requirements: Zoom on cellphone, exam on computer

See Information on Canvas/Assignments/Exams

Poll: Preferred Exam Mode

A in person
B online
C undecided

Last time

Solar Corona ("crown," not virus!): region above photosphere

- hot! $T_{\mathrm{Corona}} \approx 2$ MK $\sim$ T$_{\mathrm{center}}/8$
- very low density: $\rho_{\mathrm{corona}} \sim 10^{-16}$ g/cm^3 $\sim 10^{-18}\rho_{\mathrm{center}}$

so nuclear reactions rate per volume $r \propto \rho^2 T^4$ very tiny!

Net effect on Sun's luminosity:

none! absorbs energy from photosphere

(somehow! still an open question–"colronal heating problem")

then re-radiates – no net source or sink of solar luminsoity

Parker Solar Probe now in flight

elliptical orbit dives to $0.16au$ – directly into the corona!

Main Sequence Evolution: Recap

Main sequence ultrafast review:

- main sequence powered by *nuclear fusion in core*
- nuclear reaction net effect $4p + 2e^- \to {}^4\mathrm{He} + 2\nu_e$ escape
- so *number N of gas particles in core decreases*

  while average gas particle mass m_g increases

But for ideal gas, pressure in core

$$P_\mathrm{ideal} = \frac{N\,kT}{V} = n\ kT = \frac{\rho\,kT}{m_\mathrm{g}} \tag{1}$$

so if core T and V fixed, then P would decrease!

Q: but what really happens?

nuclear reactions reduce particle number in star core
this reduces pressure if core T and V fixed

but to keep hydrostatic equilibrium: core gets hotter!
Virial theorem say mean star temperature

$$\langle kT \rangle \sim \frac{GMm_{\rm p}}{R} \tag{2}$$

core density and temperature increases on main sequence
this leads faster nuclear reactions since $\mathcal{L}_{pp} \propto \rho^2 T^4$
which generates more energy
and leads to *main sequence brightening*
with consequences for past and future Earthlings

Here endeth the material for the Midterm Exam

All Good Things...

but this process of core burning hotter and brighter is not forever!

Q: when does main sequence end?

Main Sequence Evolution: The Last Gasps

Main sequence hydrogen burning $4p + 2e^- \rightarrow {}^4\text{He} + 2\nu_e$
required hydrogen exists in star core

but main sequence burning constantly removes hydrogen!

Sun age t	core H mass fraction $X_{\text{H,core}}$	core He mass fraction
initial: 0 Gyr	70%	28%
today: 4.55 Gyr	36%	62%

Lesson: the core already has less H than He by mass!
Sun's fuel gauge is 1/2 of its initial value!
$\rightarrow$ the Sun is middle aged!

Q: what happens when Sun's core runs out of hydrogen entirely?

Stars Under Pressure

core hydrogen burning ends when fuel supply consumed
that is, when core hydrogen exhausted: $X_{\text{core}} \to 0$
and turned into helium "ash"

with no nuclear reactions, core enclosed luminosity $l_{\text{core}} \to 0$
equilibrium lost!

- heat in core still diffuses/random walks outwards
- loses thermal pressure
- gravitation drives contraction $\to$ compression

re-pressurizes core

- ...but core heat continues to diffuse out!
  repeat cycle: continue to contract!

∞

Q: what finally stops this process?

Halting Contraction

after core hydrogen burning stops
contract to ever higher density and temperature
until/unless some new form of pressure emerges

several factors are possible

- Coulomb repulsion between the charged particles
- if hot enough and dense enough, helium "ash" can "ignite" and begin as fuel for new phase of nuclear burning
- some new effect becomes important in the core matter

Poll: Fate of the Future Sun

Vote your conscience! *All answers get credit*

When the Sun's core hydrogen is exhausted:

What's the main thing eventually stopping core contraction?

A Coulomb repulsion between charged particles

B core helium begins nuclear fusion

C some new effect becomes important in matter of the core

Losing Idealism

in fact, all above effects are important for some stars
and all have significance for the future Sun

but the dominant thing arresting core contraction after main sequence
is that the *matter in the core (*^{4}He *nuclei and electrons* e^-*)*
stops behaving as a classical ideal gas!
and a new pressure source emerges!

the origin of this transition
is due to the quantum nature of matter
and quantum effects at high density

To understand these effects we must visit the quantum world

Matter and Light: Old School

pre-quantum ("**classical**") picture of matter and light

- *matter* composed of *particles*
  with definite positions and momenta

- *light* composed of *waves*
  with definite wavelengths, but not localized positions
  and that can diffract and interfere

these familiar distinctions fell apart
in earth 20th century

The Quantum Duality of Matter and Light

early 20th century: quantum revolution

• *matter sometimes shows wave properties!*
for particle of mass m, momentum $p = mv$: **de Broglie wavelength** is

$$\lambda_{\text{deB}} = \frac{h}{p} = \frac{h}{mv} \tag{3}$$

with Planck's constant

$$h = 6.626 \times 10^{-34} \text{ Joule sec} = 4.14 \times 10^{-15} \text{ eV sec}$$

• *light sometimes acts as particle: photon* $m_\gamma = 0$
for wavelength λ, photon momentum and energy are

$$p_\gamma = \frac{h}{\lambda} \quad E_\gamma = cp_\gamma = \frac{hc}{\lambda} \tag{4}$$

Q: so what's really going on?

Is it really a particle or a wave?

So both light and matter can sometimes show wave properties and other times show particle properties

what you see depends on the experiment
a full discussion of which is the fun of taking Quantum Mechanics

for our purposes, roughly speaking:
when matter in a star is not too dense
that is, when particle spacing is more than de Broglie $\lambda_{\rm deB}$
then quantum effects not important
matter acts like ordinary ("classical") gas of particles
$\rightarrow$ *ordinary classical ideal gas approximation is good!*

but what happens in relativistic case (photons)?
and what happens when matter becomes very dense?

Uncertainty Principle

Heisenberg: wave-particle duality means

- cannot know position better than $\sim$ de Broglie wavelength
  position uncertainty $\Delta x \gtrsim \lambda_{\mathrm{deB}} \sim h/p_x$
- cannot know momentum for particle confined to Δx
  better than x-momentum uncertainty $\Delta p_x \gtrsim h/\Delta x$

can show in general: **uncertainty principle**

$$\Delta x \; \Delta p_x \; \geq \; h \tag{5}$$

as similarly

$$\Delta y \; \Delta p_y \; \geq \; h \quad \Delta z \; \Delta p_z \; \geq \; h \tag{6}$$

Compare with Newtonian ("classical") physics
particle position and momentum can be perfectly known
so classically: $\Delta x^{\mathrm{cl}} = 0$ and $\Delta p_x^{\mathrm{cl}} = 0$

Uncertainty principle has huge implications
the more tightly particles are confined (small Δx),
the higher the associated momenta and energies

Thermal Gas of Photons

important special case: **thermodynamic equilibrium**
where massive particles and photons interact
until all have thermal distributions with same temperature T

consider a thermal bath of *photons*
- temperature T
- particle mass $m_\gamma = 0$
  which means particle energy $E = \sqrt{(mc^2)^2 + (cp)^2} = cp$
  and also that $v = p/E = c$ ("relativistic particles")

want to know: what is distribution of photon wavelengths/energies
and overall average energy density and pressure

full derivation: Director's Cut Extras (next time)
quick and dirty: *dimensional analysis*

Thermal Photons: Dimensional Analysis

for a bath of photon T, we want:

- number density n_γ
- energy density ε_γ
- pressure P_γ

scales in the problem:

- thermal energy kT
- $m = 0$ so no mc^2 rest energy
  but special relativity relevant: c
- quantum effects relevant: h

Q: how to construct number density n_γ, ε_γ, P_γ?

Hint: $hc \approx 1000$ MeV fm has dimensions [energy × length]

Hint: we already have discussed this situation weeks ago!

Q: what is the name for a thermal distribution of photons?

Thermal Photons: Blackbody Radiation

dimensional analysis: kT, h, c form one *length*

$$\ell = \frac{hc}{kT} = \frac{h}{p_T} \tag{7}$$

the *thermal de Broglie length*

from this we estimate **number density**

$$n_\gamma \sim \ell^{-3} \sim \left(\frac{kT}{hc}\right)^3 \tag{8}$$

energy density

$$\varepsilon_\gamma \sim kT\ell^{-3} \sim \frac{(kT)^4}{(h^3c^3)} \tag{9}$$

pressure has dimensions of energy density, so

$$P_\gamma \sim \varepsilon \tag{10}$$

of course we know thermal photons result: blackbody radiation!

Blackbody Radiation: Exact Results

for blackbody photons at T, with $g = 2$ polarizations:

$$
\begin{aligned}
n_\gamma &= g\frac{\zeta(3)}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3 \propto T^3 \\
\varepsilon_\gamma &= g\frac{\pi^2}{30}\frac{(kT)^4}{(\hbar c)^3} = a_{\rm SB}\,T^4 \\
P_\gamma &= \frac{1}{3}\varepsilon_\gamma \propto T^4
\end{aligned}
$$

where $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3 = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots = 1.20206\ldots$

and $\hbar = h/2\pi$ is the chic "reduced Planck's constant"

and $a_{\rm SB} = \pi^2/15\ k^4/\hbar^3 c^3$ is the Stefan-Boltzmann radiation constant

Note: already saw that relativistic gas has $P = \varepsilon/3$

also note: energy flux is roughly $F \sim c\varepsilon \sim T^4$

which is Stefan-Boltzmann result!

Quantum Matter and Density

now consider a gas of *non-relativistic matter*
allow quantum effects

non-relativistic: must have $v \ll c$
so for thermal particles, typical kinetic energy $mv^2/2 \sim kT \ll mc^2$

for non-relativistic particles of mass m, at temperature T
typical kinetic energy

$$E_{\mathsf{k}} = \frac{p^2}{2m} \sim kT \tag{11}$$

gives typical *thermal momentum* $p_T \sim \sqrt{m\,kT}$

Q: what is thermal de Broglie wavelength here?
Q: estimate of number density n? mass density ρ?

thermal momentum $p_T \sim \sqrt{m\,kT}$

gives **thermal de Broglie wavelength**

$$\lambda_{\mathrm{deB}}(T) = \frac{h}{p_T} \sim \left(\frac{h^2}{m\,kT}\right)^{1/2} \tag{12}$$

and so naively expect a number density

$$n_{\mathrm{naive}}(T) \sim \lambda_{\mathrm{deB}}(T)^{-3} \sim \left(\frac{mkT}{\hbar^2}\right)^{3/2} \tag{13}$$

and mass density

$$\rho_{\mathrm{naive}}(T) = m\; n_{\mathrm{naive}}(T) \sim m \left(\frac{mkT}{\hbar^2}\right)^{3/2} \tag{14}$$

for a given species m, this gives a number density $n(T)$ entirely and universally determined by temperature!

21

Q: what is strange about this result?

Hint: what sets $\rho(T)$? apply to objects in this room?

naively expect mass density

$$\rho_{\text{naive}}(T) = m\ n_{\text{naive}}(T) \sim m \left(\frac{mkT}{\hbar^2}\right)^{3/2} \tag{15}$$

but that can't be right!

density of water in you, a beverage, and the air are all different!

also: for $T = 300$ K this gives $n_{\text{naive,water}} \sim 10^{27}\ \text{cm}^{-3}$, and $\rho_{\text{naive,water}} \sim 3 \times 10^4\ \text{g/cm}^3$. *Yikes!*

Q: where did we go wrong?

really: we have assumed particle spacing always around $\lambda_{\text{deB}}(T)$
this is "quantum size" of thermal particles

this sets a special density: the **quantum concentration**

$$n_Q = \left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} \sim \frac{1}{\lambda_{\text{deB}}^3} \tag{16}$$

n_Q rises with T since $\lambda_{\text{deB}}(T) = h/p_T \propto T^{-1/2}$

but clearly

- real particle density can be lower or higher!
- n_Q is high compared to everyday matter

Q: why do we expect physically if $n \ll n_Q$? if $n \gtrsim n_Q$?

• if $n \ll n_Q$:
particle spacings larger than thermal de Broglie wavelength
particles are "too far apart" for quantum effects
expectation:
quantum effects small: ordinary ("classical") ideal gas!

• if $n \gtrsim n_Q$:
particle spacings of same order as de Broglie
now expect departures from classical ideal gas
must include quantum effects

namely: combine Pauli exclusion principle
with Heisenberg uncertainty principle

Identical Particles

experiments and theory show: *all particles of each species are completely identical and* **indistinguishable**

example: all electrons are completely identical
as are all photons, neutrons, protons, etc
always have exactly same charge, mass, spin

spoiler: not just result of a high-quality "electron factory"
but really: space filled with "electron field"
whose quantum excitations are electron particles

Pauli: this has profound effects in quantum mechanics for systems of multiple particles

Q: experts–what's the rule? what does it depend on?

Pauli Principle

behavior of identical particles
depends on spin (particles "self" angular momentum)

Bosons: particles with spin $S = 0, 1, 2, \ldots$
example: photon $S = 1$ is a boson
no restriction on number of boson in same quantum state
"bosons are social" – party anmials of the quantum world

Fermions: spin $S = 1/2, 3/2, 5/2, \ldots$ ("half-integer spin")
ex: electrons, protons, neutrons all are $S = 1/2$ Fermions
at most one Fermion per quantum state
"fermions are loners" – they want to be alone!
Pauli exclusion principle

profound implications for the nature of matter!

Uncertainty Principle

Heisenberg: wave-particle duality means

- cannot know position better than $\sim$ de Broglie wavelength position uncertainty $\Delta x \gtrsim \lambda_{\mathrm{deB}} \sim h/p_x$
- cannot know momentum for particle confined to Δx better than x-momentum uncertainty $\Delta p_x \gtrsim h/\Delta x$

can show in general: **uncertainty principle**

$$\Delta x \ \Delta p_x \ \geq \ h \tag{17}$$

so in volume $\Delta V = \Delta x \ \Delta y \ \Delta z$

$$\Delta V \ \Delta^3 p \ \geq \ h^3 \tag{18}$$

with "momentum space" volume $\Delta^3 p = \Delta p_x \ \Delta p_y \ \Delta p_z$

Q: so what is maximum number density for gas of electrons?

Maximum Fermion Density

Pauli exclusion principle means fermions obey

$$\Delta V \ \Delta^3 p \ \geq \ h^3 \tag{19}$$

so for gas of electrons with $S = 1/2$

- 2 possible spin states $(\uparrow, \downarrow)$
  same energy in both: *degenerate* states
- maximum number density n_e set by

$$n_{e,\mathrm{max}} \ \Delta V = 2 \tag{20}$$

which gives

$$n_{e,\mathrm{max}} = \frac{2}{\Delta V} = \sum_p \frac{2}{h^3} \Delta^3 p \tag{21}$$

momentum space volume $4\pi/3 \ p^3$ has $\Delta^3 p = 4\pi p^2 \ dp$
up to some *maximum momentum* ("Fermi momentum") p_{Fermi}

Pauli-approved maximum electron density
sums (integrates) all possible momenta up to some $p_{\rm Fermi}$

$$n_{e,\rm max} = \frac{2}{\Delta V} = \frac{2}{h^3} 4\pi \int_0^{p_{\rm Fermi}} p^2 \, dp \tag{22}$$

$$= \frac{8\pi}{3h^3} \, p_{\rm Fermi}^3 \tag{23}$$

maximum density also called *degenerate number density*

required maximum momentum to have number density n_e:

$$p_{\rm Fermi} = \left(\frac{3n_e}{8\pi}\right)^{1/3} h \sim \frac{h}{\ell} \tag{24}$$

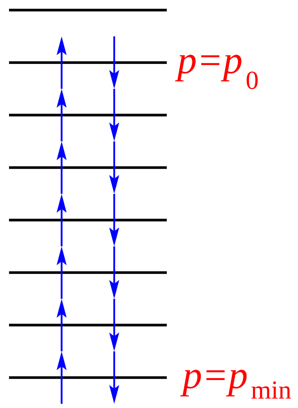
so Fermi momentum set by uncertainty principle $p_{\rm Fermi} \, \ell \sim h$
where distance $\ell = n_e^{-1/3}$ is *typical particle spacing*

Q: for degenerate gas, what is special about states above, below $p_{\rm Fermi}$*?*

Fermi momentum for electron gas of number density n_e:

$$p_{\text{Fermi}} = \left(\frac{3n_e}{8\pi}\right)^{1/3} h \;\sim\; \frac{h}{\ell} \tag{25}$$

number density n_e sets highest momentum reached by filling all states up to p_{Fermi} and leaving all others empty



now consider the case of $p_T = \sqrt{mkT} \gg p_{\text{Fermi}}$

Q: what does this mean physically?

Q: what does this mean for density?

if gas is completely degenerate

$$p^3_{\text{Fermi}} = \frac{3n_e h^3}{8\pi} \tag{26}$$

so so if $p_{\text{Fermi}} \ll p_T = \sqrt{mkT}$, then physically
thermally available momentum states far exceed needed p_0
momentum states don't have to be "packed full"
density is not maximal $\rightarrow$ *gas is not degenerate*

quantitatively, we have

$$\frac{3n_e h^3}{8\pi} \ll p_T^3 = (mkT)^{3/2} \tag{27}$$

$$n_e \ll \frac{8\pi}{3}\left(\frac{mkT}{4\pi^2\hbar^2}\right)^{3/2} \sim n_Q \tag{28}$$

lesson: *non-degenerate* $\Leftrightarrow$ density $\ll$ quantum concentration

so air in this room, gas in solar core today: non-degenerate