Astro 404

Lecture 21

Oct. 11, 2021

Announcements:

- Good news: *no homework due Friday!*
- *Bad news:* **Hour Exam Friday Oct 18.** Info on Canvas
  all homework solutions are posted
- Overview today, Review on Wednesday

Last time: End of main sequence

- *core density and temperature increases on main sequence*
- but at end of main sequence equilibrium lost!
  core contracts until new pressure source emerges

Conclusion:

need to understand matter at high density and pressure

quantum effects become important

Quantum Mechanics – Highlights

matter and light: can show particle=like proprties
or wave-like properties depending on the experiment

de Broglie wavelength

for particle of mass m and momentum p

$$\lambda_{\mathrm{deB}} = \frac{h}{p} = \frac{h}{mv} \tag{1}$$

expect wave-ke behavior when particle confined or interacts
on scale $\leq \lambda_{\mathrm{deB}}$

uncertainty principle

$$\Delta x \;, \Delta p \;\geq\; \frac{1}{2}\,\frac{h}{2\pi} = \frac{1}{2}\hbar \tag{2}$$

Heisenberg: wave-particle duality means
- cannot know position better than $\sim$ de Broglie wavelength

position uncertainty $\Delta x \gtrsim \lambda_{\mathsf{deB}} \sim h/p_x$

- cannot know momentum for particle confined to Δx better than x-momentum uncertainty $\Delta p_x \gtrsim h/\Delta x$

A Quantum Baseball?

regulation mass $m = 5$ oz $= 0.14$ kg
easy toss: $v \sim 1$ m/s
$\rightarrow$ momentum $p = mv \sim 0.14$ kg m/s
$\rightarrow$ de Broglie wavelength

$$\lambda_{\text{deB,baseball}} = \frac{h}{p} = 5 \times 10^{-33} \text{ m} \lesssim 10^{-14} \times \text{size of proton} \quad (3)$$

wave properties and hence quantum effects unobservably small!
$\rightarrow$ expect baseballs to exhibit classical (Newtonian) behavior
$\rightarrow$ can't blame fielding errors on quantum mechanics!

Q: in what circumstances would quantum effects not be small?
i.e., for what objects is λ_{deB} larger?

Identical Particles

experiments and theory show: *all particles of each species are completely identical and* **indistinguishable**

example: all electrons are completely identical
as are all photons, neutrons, protons, etc
always have exactly same charge, mass, spin

spoiler: not just result of a high-quality "electron factory"
but really: space filled with "electron field"
whose quantum excitations are electron particles

Pauli: this has profound effects in quantum mechanics for systems of multiple particles

Q: experts–what's the rule? what does it depend on?

Pauli Principle

behavior of identical particles
depends on spin (particles "self" angular momentum)

Bosons: particles with spin $S = 0, 1, 2, \ldots$
example: photon $S = 1$ is a boson
no restriction on number of boson in same quantum state
"bosons are social" – party anmials of the quantum world

Fermions: spin $S = 1/2, 3/2, 5/2, \ldots$ ("half-integer spin")
ex: electrons, protons, neutrons all are $S = 1/2$ Fermions
at most one Fermion per quantum state
"fermions are loners" – they want to be alone!
Pauli exclusion principle

profound implications for the nature of matter!

The Quantum Atom: Bohr Model

In stellar interiors, atoms are mostly ionized
electrons torn from nuclei, move freely

but atom properties critical in stellar atomospheres
and are a "familiar" illustration of quantum effects

Bohr Model:
quantum structure of atom: e orbits are matter waves
"semiclassical"–mixes Newtonian & quantum ideas

- de Broglie waves $\rightarrow$ standing waves in atom
- e orbits circular
- only certain radii, speeds allowed ("quantized states")
  $\rightarrow$ only certain allowed energies
- during e transitions between states, photon emitted
  $\rightarrow$ photon energies quantized $\rightarrow$ spectral lines

Bohr Atom: Quantum Electrons Orbit Nucleus

Ingredients:

- circular orbits
- electrons have de Broglie wavelengths $\lambda = h/p = h/m_e v$

- standing waves:

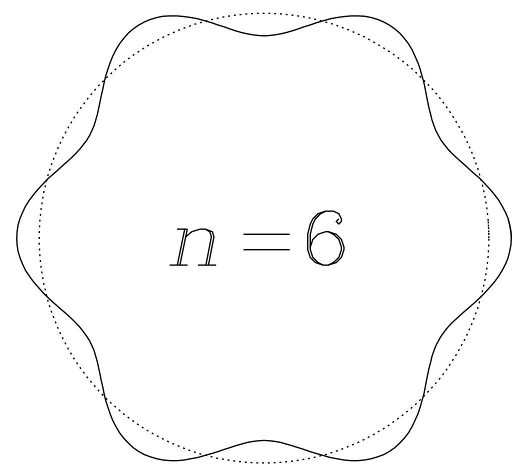
Demo: slinky

e orbit path length

an integer multiple of λ:

$$2\pi r = n\lambda = n\frac{h}{m_e v} \tag{4}$$

$\rightarrow$ for each n, radii and speeds related



- Coulomb force provides centripetal acceleration

Coulomb force: electrical attraction between opposite charges
an inverse square law! same structure as gravity!
For nucleus of charge $q_1 = Ze$ and electrons of charge $q_2 = -e$
magntude of force is

$$F_{\text{Coulomb}} = \frac{q_1 q_2}{r^2} = \frac{Ze^2}{r^2} \tag{5}$$

(cgs charge units: $e_{\text{cgs}}^2 = ke_{\text{SI}}^2 = e_{\text{SI}}^2/4\pi\varepsilon_0$)

Coulomb provides electron's centripetal acceleration:

$$m_e a_c = F_{\text{Coulomb}} \tag{6}$$

$$m_e \frac{v^2}{r} = \frac{Ze^2}{r^2} \tag{7}$$

another relation between r and v
→ two equations, two unknowns → solution exists

Bohr: fit *integer* number $n \geq 1$ *standing waves*
into Coulomb-controlled circular orbits

$\Rightarrow$ only certain definite radii/speeds/momenta/energies allowed

$\Rightarrow$ "quantized" orbits • allowed radii:

$$r_n = n^2 \frac{\hbar^2}{Ze^2 m_e} \tag{8}$$

• allowed speeds:

$$v_n = \frac{1}{n} \frac{Ze^2}{\hbar} \tag{9}$$

• potential, kinetic, and total energy:

$$E_{n,\mathrm{pot}} = -\frac{Ze^2}{r_n} = -\frac{Z^2 e^4 m_e^2}{\hbar^2} \frac{1}{n^2} \tag{10}$$

$$E_{n,\mathrm{kin}} = \frac{1}{2} m_e v_n^2 = \frac{Z^2 e^4 m_e^2}{2\hbar^2} \frac{1}{n^2} = -\frac{1}{2} E_{n,\mathrm{pot}} \tag{11}$$

$$E_{n\,\mathrm{tot}} = E_{n,\mathrm{pot}} + E_{n,\mathrm{kin}} = -\frac{Z^2 e^4 m_e^2}{w\hbar^2} \frac{1}{n^2} \tag{12}$$

Q: what happens as n increases?

allowed radii:

$$r_{n} = n^2 \frac{\hbar^2}{Ze^2 m_e} \tag{13}$$

allowed speeds:

$$v_{n} = \frac{1}{n} \frac{Ze^2}{\hbar} \tag{14}$$

potential, kinetic, and total energy:

$$\begin{aligned}
E_{n,\text{pot}} &= -\frac{Ze^2}{r_n} = -\frac{Z^2 e^4 m_e^2}{\hbar^2} \frac{1}{n^2} && (15)\\
E_{n,\text{kin}} &= \frac{1}{2} m_e v_n^2 == \frac{Z^2 e^4 m_e^2}{2\hbar^2} \frac{1}{n^2} = -\frac{1}{2} E_{n,\text{pot}} && (16)\\
E_{n\,\text{tot}} &= E_{n,\text{pot}} + E_{n,\text{kin}} = -\frac{Z^2 e^4 m_e^2}{w\hbar^2} \frac{1}{n^2} && (17)
\end{aligned}$$

consequences of quantum de Broglie waves

- only certain specific ("discrete") orbits allowed
- infinitely many possible bound energy states

that is, $E_{n,\text{tot}} < 0$ for any $n < \infty$

- but there is one unique lowest energy state
  namely: $n = 1$, the "**ground state**"
  also lowest v_n, $p_n = m_e v_n$
- still Virial-type relation $E_{n,\text{kin}} = -E_{\text{pot}}/2$!

Building Quantum Atoms

Bohr model: excellent (but not perfect!) description of hydrogen
for more complex atoms with > 1 electron:

- electrons interact with each other
- energy level structure more complicated

but while these details change, still find

- only certain allwed momentum and energy states
  "discrete" spectrum of states
  `www: examples of discrete lines in atomic spectra`
- there are infinitely many possible bound states
- there is one unique ground state of lowest energy

To "build" an atom with Z electtrons
have to fill the available states with them

Q: how to do this with the lowest possible energy?

Filing Levels: How Low Can You Go?

Bohr model example: energy states

$$
\begin{aligned}
E_{n\,\mathrm{tot}} &= -\frac{Z^2 e^4 m_e^2}{w\hbar^2}\frac{1}{n^2} && (18)\\
&= \frac{E_1}{n^2} && (19)\\
&= -13.6\ \mathrm{eV}\frac{Z^2}{n^2} && (20)
\end{aligned}
$$

ground state: $n = 1$ has $E_1 = -13.6$ eV for hydrogen

tempting: put all Z electrons in ground state!?!?

but this is illegal!

Pauli says: electrons have spin 1/2, and are Fermions
so only 2 per state are allowe

Pauli demands that electrons fill beyond ground state!

Q: how to do this with minimum energy?

Q: what sets highest energy level?

Building Quantum Systems: Pauli Rules

central result of quantum mechanics:
when quantum particles confined to finite volume of space
not all energies are allowed!

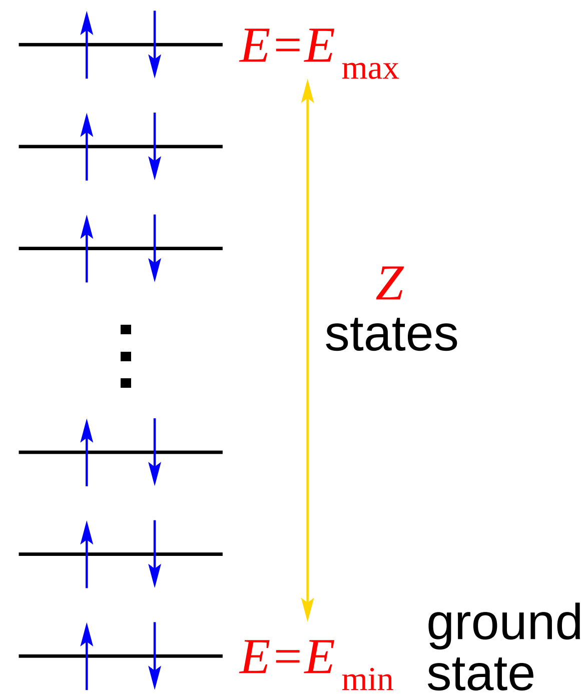
allowed states have definite energies: "energy levels"
which may or may not be different for different spin states

Pauli Principle: at most one Fermion per quantum state
including both *energy* and spin
if energy levels the same for spin up and down
then two particles per energy level
for lowest-energy filling, highest level set by number of particles

Building an Atom

to "build" an atom:

- lowest energy level: **ground state**
  fits up to 2 electrons, spins ↑↓
  these have same energy: "degenerate"
- for normal (unexcited) atom:
  keep adding electrons
  two per energy level
  from the lowest available energy up
- after ground state, fill first excited state
- repeat until all Z electrons added
-] highest level set by number of e



note that electrons in the highest levels
have largest energy, highest speeds and momenta least bound!

Dense Stars as Quantum Gasses

We have seen: after the Main Sequence
stellar interiors become very dense

Dense interior regions of stars:

- still are gasses of free particles: nuclei and electrons
- but so crowded that their de Broglie wavelengths can overlap!

So we will model a dense stellar interor as
an ideal quantum gas
also known as a *Fermi gas*

- free particles (ignore interactions such as repulsion)
- but quantum properties, obeying Pauli principle
- bound by the star's gravity

Quantum Matter and Density

now consider a gas of *non-relativistic matter*
allow quantum effects

non-relativistic: must have $v \ll c$
so for thermal particles, typical kinetic energy $mv^2/2 \sim kT \ll mc^2$

for non-relativistic particles of mass m, at temperature T
typical kinetic energy

$$E_{\mathrm{k}} = \frac{p^2}{2m} \sim kT \tag{21}$$

gives typical *thermal momentum* $p_T \sim \sqrt{m\,kT}$

Q: what is thermal de Broglie wavelength here?
Q: estimate of number density n? mass density ρ?

thermal momentum $p_T \sim \sqrt{m\,kT}$
gives **thermal de Broglie wavelength**

$$\lambda_{\rm deB}(T) = \frac{h}{p_T} \sim \left(\frac{h^2}{m\,kT}\right)^{1/2} \tag{22}$$

and so naively expect a number density

$$n_{\rm naive}(T) \sim \lambda_{\rm deB}(T)^{-3} \sim \left(\frac{mkT}{\hbar^2}\right)^{3/2} \tag{23}$$

and mass density

$$\rho_{\rm naive}(T) = m\; n_{\rm naive}(T) \sim m \left(\frac{mkT}{\hbar^2}\right)^{3/2} \tag{24}$$

for a given species m, this gives a number density $n(T)$
entirely and universally determined by temperature!

Q: what is strange about this result?
Hint: what sets $\rho(T)$? apply to objects in this room?

naively expect mass density

$$\rho_{\text{naive}}(T) = m \; n_{\text{naive}}(T) \sim m \left(\frac{mkT}{\hbar^2}\right)^{3/2} \tag{25}$$

but that can't be right!
density of water in you, a beverage, and the air
are all different!

also: for $T = 300$ K this gives
$n_{\text{naive,water}} \sim 10^{27}\ \text{cm}^{-3}$, and $\rho_{\text{naive,water}} \sim 3\times10^4\ \text{g/cm}^3$. *Yikes!*

Q: where did we go wrong?

really: we have assumed particle spacing always around $\lambda_{\rm deB}(T)$
this is "quantum size" of thermal particles

this sets a special density: the **quantum concentration**

$$n_Q = \left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} \sim \frac{1}{\lambda_{\rm deB}^3} \qquad (26)$$

n_Q rises with T since $\lambda_{\rm deB}(T) = h/p_T \propto T^{-1/2}$

but clearly

- real particle density can be lower or higher!
- n_Q is high compared to everyday matter

Q: why do we expect physically if $n \ll n_Q$? if $n \gtrsim n_Q$?

• if $n \ll n_Q$:
particle spacings larger than thermal de Broglie wavelength
particles are "too far apart" for quantum effects
expectation:
quantum effects small: ordinary ("classical") ideal gas!

• if $n \gtrsim n_Q$:
particle spacings of same order as de Broglie
now expect departures from classical ideal gas
must include quantum effects

namely: combine Pauli exclusion principle
with Heisenberg uncertainty principle

Uncertainty Principle Revisited

uncertainty principle: for motion in one dimension

$$\Delta x \; \Delta p_x \; \geq \; h \tag{27}$$

so in volume $\Delta V = \Delta x \; \Delta y \; \Delta z$

$$\Delta V \; \Delta^3 p \; \geq \; h^3 \tag{28}$$

with "momentum space" volume $\Delta^3 p = \Delta p_x \; \Delta p_y \; \Delta p_z$

Q: so what is maximum number density for gas of electrons?

Maximum Fermion Density

Pauli exclusion principle means fermions obey

$$\Delta V \; \Delta^3 p \; \geq \; h^3 \tag{29}$$

so for gas of electrons with $S = 1/2$

- 2 possible spin states $(\uparrow, \downarrow)$
  same energy in both: *degenerate* states
- maximum number density n_e set by

$$n_{e,\mathrm{max}} \; \Delta V = 2 \tag{30}$$

which gives

$$n_{e,\mathrm{max}} = \frac{2}{\Delta V} = \sum_p \frac{2}{h^3} \Delta^3 p \tag{31}$$

momentum space volume $4\pi/3 \; p^3$ has $\Delta^3 p = 4\pi p^2 \; dp$
up to some *maximum momentum* ("Fermi momentum") p_{Fermi}

Pauli-approved maximum electron density

sums (integrates) all possible momenta up to some $p_{\rm Fermi}$

$$n_{e,\rm max} = \frac{2}{\Delta V} = \frac{2}{h^3} 4\pi \int_0^{p_{\rm Fermi}} p^2 \; dp \tag{32}$$

$$= \frac{8\pi}{3h^3} \; p_{\rm Fermi}^3 \tag{33}$$

maximum density also called *degenerate number density*

required maximum momentum to have number density n_e:

$$p_{\rm Fermi} = \left(\frac{3n_e}{8\pi}\right)^{1/3} h \; \sim \; \frac{h}{\ell} \tag{34}$$

so Fermi momentum set by uncertainty principle $p_{\rm Fermi}\, \ell \sim h$

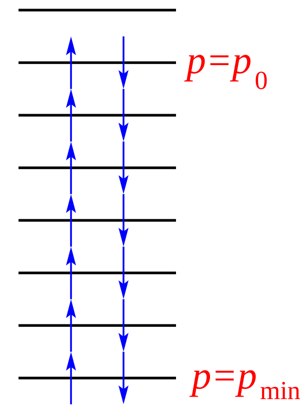
where distance $\ell = n_e^{-1/3}$ is *typical particle spacing*

23 *Q: for degenerate gas, what is special about states above, below* $p_{\rm Fermi}$*?*

Fermi momentum for electron gas of number density n_e:

$$p_{\text{Fermi}} = \left(\frac{3n_e}{8\pi}\right)^{1/3} h \ \sim \ \frac{h}{\ell} \tag{35}$$

number density n_e sets highest momentum reached by filling all states up to p_{Fermi} and leaving all others empty



now consider the case of $p_T = \sqrt{mkT} \gg p_{\text{Fermi}}$

Q: what does this mean physically?

Q: what does this mean for density?

if gas is completely degenerate

$$p_{\text{Fermi}}^3 = \frac{3n_e h^3}{8\pi} \tag{36}$$

so so if $p_{\text{Fermi}} \ll p_T = \sqrt{mkT}$, then physically
thermally available momentum states far exceed needed p_0
momentum states don't have to be "packed full"
density is not maximal $\rightarrow$ *gas is not degenerate*

quantitatively, we have

$$\frac{3n_e h^3}{8\pi} \ll p_T^3 = (mkT)^{3/2} \tag{37}$$

$$n_e \ll \frac{8\pi}{3}\left(\frac{mkT}{4\pi^2\hbar^2}\right)^{3/2} \sim n_Q \tag{38}$$

lesson: *non-degenerate* $\Leftrightarrow$ density $\ll$ quantum concentration

so air in this room, gas in solar core today: non-degenerate