Astro 404
Lecture 22
Oct. 13, 2021

Announcements:

- Good news: *no homework due Friday!*
- *Bad news:* **Hour Exam Friday Oct 18.** Info on Canvas
  all homework solutions are posted
  review today as part of class

Last time:
building technology to understand matter at high pressure
to use for understanding extreme conditions in stars
spoiler: high-density leads to high pressure
  not due to thermal motions
  but reflecting uncertainty principle + Pauli principle
  dense stars are supported by quantum mechanics!

Atoms as Quantum Objects

in atoms: quantum effects critical
electrons show wave-like behavior: position not well defined!
only discrete radii allowed as standing waves

electron motion only occurs in specific "states"
each with its own energy

- lowest energy: ground state
- infinitely many levels with higher energy

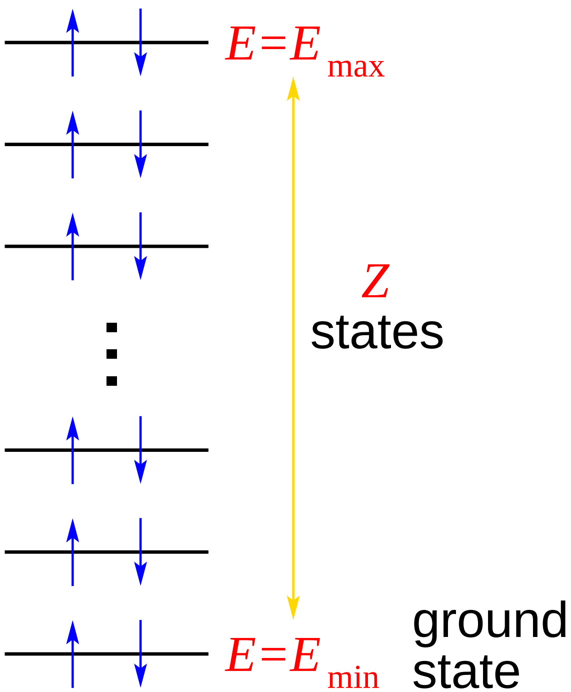
building atoms: for neutral atom with Z protons
add Z electrons to fill energy level
Pauli principle: only one electron per state (energy + spin)
so at most 2 electrons per energy level: $\uparrow\downarrow$

Q: how to "build" a normal (unexcited) atom?

Building an Atom

to "build" an atom:

- lowest energy level: **ground state**
  fits up to 2 electrons, spins $\uparrow\downarrow$
  these have same energy: "degenerate"
- for normal (unexcited) atom:
  keep adding electrons
  two per energy level
  from the lowest available energy up
- after ground state, fill first excited state
- repeat until all Z electrons added



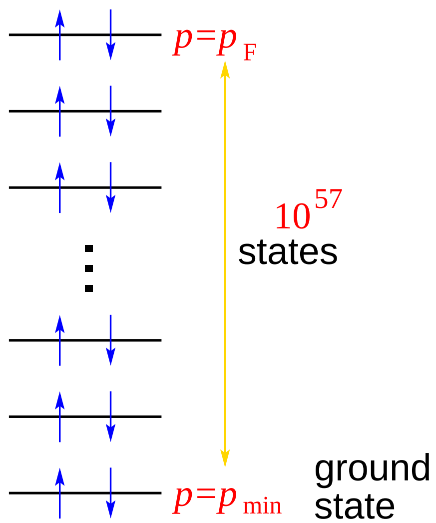
compare and contrast to degenerate star with $\sim 10^{57}$ atoms

Q: what confines the electrons? what sets levels? how to fill?

A Degenerate Electron Gas

consider a cold degenerate star: a $N \approx 10^{57}$ particle gas of free particles at $T = 0$, in volume confined by gravity!

free quantum particles: states labeled by *de Broglie wavelength* $\lambda = h/p$ or equally well by *momentum* p or energy $E(p)$



to build star out of degenerate electron gas:

- start in ground state adding 2 electrons $\uparrow\downarrow$
- fill first excited state, etc
- until reach highest needed level: Fermi level labeled by Fermi momentum p_{Fermi} or Fermi energy $E(p_{\mathrm{Fermi}})$ highest level reached when fill up from ground state
- also maximal energy of Fermi gas particle: $E_{\mathrm{max}} = E_F$

Fermi Gas: Counting States

Fermi gas: non-interacting quantum fermion particles
inside "box" of volume volume $V = L^3$

as with Bohr model: quantum motion $\rightarrow$ standing waves:
in 3D, waves avoid box edge $\sin(k_x x)\ \sin(k_y y)\ \sin(k_z z)$ with

$$\vec{k} = (k_x, k_y, k_z) = \frac{\pi}{L}(n_x, n_x, n_z) \tag{1}$$

with n_x, n_y, n_z all integers ("quantum numbers")
each labels a possible state

note: wavelength is given by $2\pi x/\lambda = kx$
so $\lambda = 2\pi/k$

Counting Wave States

can use quantum numbers to count states:
using $k_x = \pi n_x / L$:

- between n_x and $n_x + dn_x$
  add $dn_x = L\ dk_x / 2\pi$ more states
- similar for y and z

so total number of states is this range is

$$dN = dn_x\ dn_y\ dn_z = \frac{L^3}{\pi^3} dk_x\, dk_y\, dk_z = \frac{V}{\pi^3} dk_x\, dk_y\, dk_z \qquad (2)$$

"k-space" contains $(L/\pi)^3$ states per volume $dk_x\, dk_y\, dk_z$

in k-space sphere, in octant where all $k_x, k_y, k_z > 0$
volume between k and $k + dk$: $dV_k = 4\pi k^2\ dk/8$

Density of States

number of quantum states between k and $k + dk$:

$$dN_k = \frac{V}{\pi^3}\, dV_k = \frac{V}{8\pi^3}\, 4\pi k^2\, dk \tag{3}$$

de Broglie: waves have $p = h/\lambda$
and since $k = 2\pi/\lambda$,

$$p = \frac{hk}{2\pi} = \hbar k \tag{4}$$

and so k space is really momentum space, and thus
number of wave states is really number of momentum states

$$dN_p = \frac{V}{h^3}\, 4\pi p^2\, dp \tag{5}$$

for electrons (or neutrons): $g_e = 2$ spin states per momentums state

$$dN_e = g_e \frac{V}{h^3}\, 4\pi p^2\, dp \tag{6}$$

A Cold Electron Gas

in a cold electron gas: all states filled from ground up
each level fully occupied by 2 electrons: ↑↓
the is the lowest energy configuration possible
known as a **degenerate** state

highest level populated is known as *Fermi level*
with energy ϵ_F and momentum p_F

number of electrons:

$$N_e = g_e \frac{V}{h^3} \int_0^{p_F} 4\pi p^2 \, dp = \frac{8\pi}{3} V \left(\frac{p_F}{h} \right)^3 \quad (7)$$

and so **number density**

$$n_e = \frac{N_e}{V} = \frac{8\pi}{3h^3} \, p_F^3 \quad (8)$$

Degeneracy and Number Density

electron Fermi gas **number density**

$$n_e = \frac{N_e}{V} = \frac{8\pi}{3h^3}\, p_F^3 \tag{9}$$

particle states filled with maximum efficiency up to p_{Fermi}

$\Rightarrow$ highest density possible for this number of electrons

so Fermi level sets maximum number density

but also works the other way

to create a degenerate gas of electrons with number density n_e
requires Fermi momentum

$$p_{\mathrm{Fermi}} = \left(\frac{3n_e}{8\pi}\right)^{1/3} h \ \sim \ \frac{h}{\ell} \tag{10}$$

where $\ell = 1/n_e^{1/3}$ is the typical electron spacing

number density n_e sets highest momentum
reached by filling all states up to p_{Fermi}
and leaving all others empty

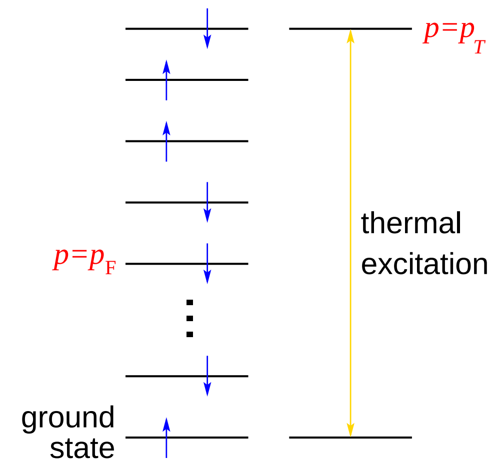
so far: Fermi gas at $T = 0$ – pretty cold!
now consider T nonzero, with

- $kT \gg E_F$, and equivalently
- $p_T = \sqrt{mkT} \gg p_{\mathrm{Fermi}}$

Q: what does this mean physically?
Q: what does this mean for density?

if gas is completely degenerate

$$p^3_{\mathrm{Fermi}} = \frac{3 n_e h^3}{8\pi}$$



so if $p_{\mathrm{Fermi}} \ll p_T = \sqrt{mkT}$, then physically
thermal excitations of momentum states
far exceed needed p_F
momentum states don't have to be "packed full"
each particle can have wide range of momenta
motion no longer dominated by quantum mechanics
density is not maximal $\rightarrow$ *gas is not degenerate*

so *heating a degenerate gas* to $kT \gg E(p_{\mathrm{Fermi}})$
"lifts" the degeneracy $\rightarrow$ recover classical ideal gas
this will be explosively crucial for the fate of Sun-like stars!

Hour Exam Review

Hour Exam Overview

Exam Information Posted on Canvas

highlights:

- Hour Exam is in two days – **Friday Oct 18**
- Time: 50 minutes, usual class time 10:00 - 10:50 am
- exam is **closed book, closed notes, closed internet**
  but you may bring *a calculator*
  and one ordinary 8.5 in $\times$ 11 in sheet of paper
  with your name and anything you can write by hand

Poll: How to Take the Exam

I plan to take the exam

A In person

B Online

C Undecided but leaning for in person

D Undecided but leaning for online

Exam Topics: Some Main Themes

star distances, fluxes, luminosity: what are they? how related? how measured with magnitudes?

HR diagram: main regions? how does it encode stellar evolution? results for star clusters vs all stars?

density, enclosed mass, gravtitional acceleration: what are they? equations?

hydrostatic equilibrium: what is it? why important? equation?

Virial theorem: what is it? when/where does it apply? main lessons?

main sequence: what controls the sequence? which stars live there?

nuclear reactions and hydrogen burning: main effect? neutrinos and their connection to energy generation?

Questions about any of these? Question about 2019 exam?