Astro 404

Lecture 23

Oct. 18, 2021

Announcements:

- **PS7 posted today, due Friday**
- Exam grading elves hard at work

Before the exam:: final piece of stellar physics puzzle

matter and radiation at high density and pressure

- matter: classical ideal gas vs quantum degenerate gas
- **ideal gas**: $P = n\ kT = \rho\, kT/m_{\rm g}$

  average particle properties

$$\langle v \rangle = \sqrt{\frac{3kT}{m_{\rm g}}}\ ,\quad \langle p \rangle = m_{\rm g} \langle v \rangle = \sqrt{3 m_{\rm g} kT}\ ,\quad \langle E_{\rm kin} \rangle = \frac{1}{2} m_{\rm g} \langle v^2 \rangle = \frac{3}{2} kT$$

Q: ideal gas at $T = 0$: particles speeds? KE? pressure?

Q: why must this be different for a quantum gas at $T = 0$?

A Cold Ideal Gas: Classical vs Quantum Properties

consider a cold gas: $T \to 0$ (absolute zero!)

Classical Ideal Gas at $T = 0$:
- random particle motion stops!
- $\langle v \rangle \to 0$, same for momentum, KE
- pressure $P \to 0$!

In the quantum picture Heisenberg: $\Delta x \, \Delta p > h$
a quantum gas confined in a finite region Δx
must have a nonzero spread in momenta $\Delta p > 0$

and so for a quantum gas
- particles have $\langle p \rangle > 0$: motion does not stop!
- also nonzero speeds, $\langle E_{\text{kin}} \rangle > 0$

Q: what sets highest particle momentum and energy?

Degenerate Quantum Gas of Electrons

electrons:

two possible spin states ↑↓ per momentum/energy level

gas:

non-interacting particles

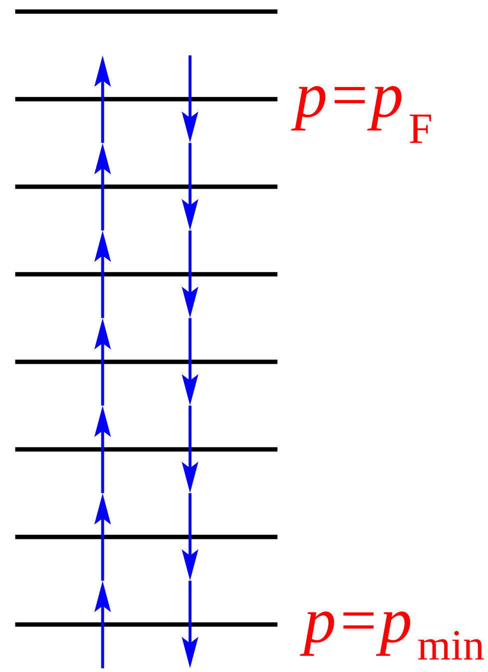
degenerate:

all momentum or energy states fully filled
from ground state upwards

until highest state:

Fermi level p_F and $E_F(p_F)$

- this is the lowest-energy configuration
- highest level p_F set by number of particles



Degenerate Quantum Gas: Density of States

number of quantum states between k and $k + dk$:

$$dN_k = \frac{V}{\pi^3} \, dV_k = \frac{V}{8\pi^3} \, 4\pi k^2 \, dk \tag{1}$$

de Broglie: waves have $p = h/\lambda$
and since $k = 2\pi/\lambda$,

$$p = \frac{hk}{2\pi} = \hbar k \tag{2}$$

and so k space is really momentum space, and thus
number of wave states is really number of momentum states

$$dN_p = \frac{V}{h^3} \, 4\pi p^2 \, dp \tag{3}$$

for electrons (or neutrons): $g_e = 2$ spin states per momentums state

$$dN_e = g_e \frac{V}{h^3} \, 4\pi p^2 \, dp \tag{4}$$

A Degenerate Electron Gas

in a cold, degenerate electron gas:
  all states filled from ground up
each level fully occupied by 2 electrons: ↑↓
the is the lowest energy configuration possible
known as a **degenerate** state

highest level populated is known as *Fermi level*
with energy ϵ_F and momentum p_F

number of electrons:

$$N_e = g_e \frac{V}{h^3} \int_0^{p_F} 4\pi p^2 \; dp = \frac{8\pi}{3} V \left(\frac{p_F}{h} \right)^3 \tag{5}$$

and so **number density**

$$n_e = \frac{N_e}{V} = \frac{8\pi}{3h^3} \; p_F^3 \tag{6}$$

Degeneracy and Number Density

electron Fermi gas **number density**

$$n_e = \frac{N_e}{V} = \frac{8\pi}{3h^3} \, p_F^3 \qquad (7)$$

particle states filled with maximum efficiency up to p_{Fermi}

Note:

- *any* number density n_e is possible
- but the higher n_e, the larger must be p_F
- denser degenerate gas $\rightarrow$ more energetic particles

so Fermi level sets maximum number density

but also works the other way

to create a degenerate gas of electrons with number density n_e
requires Fermi momentum

$$p_{\mathrm{Fermi}} = \left(\frac{3n_e}{8\pi}\right)^{1/3} h \ \sim \ \frac{h}{\ell} \tag{8}$$

where $\ell = 1/n_e^{1/3}$ is the typical electron spacing
$\rightarrow$ particles at Fermi level spaced by de Broglie wavelength h/p_F!

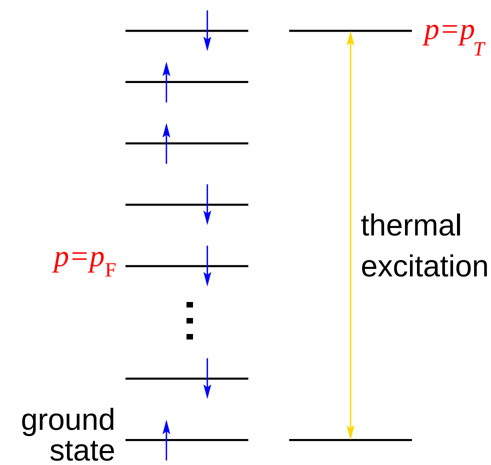
number density n_e sets highest momentum
reached by filling all states up to p_{Fermi}
and leaving all others empty

now consider T nonzero, with $p_T = \sqrt{mkT} \gg p_{\mathrm{Fermi}}$
Q: what does this mean physically?
Q: what does this mean for density?

if gas is completely degenerate

$$p_{\rm Fermi}^3 = \frac{3 n_e h^3}{8\pi}$$

so if $p_{\rm Fermi} \ll p_T = \sqrt{mkT}$, then physically
thermal excitations of momentum states
far exceed needed p_0
momentum states don't have to be "packed full"
and uncertainty principle allows larger spacing
density is not maximal → *gas is not degenerate*



so *heating a degenerate gas* to $kT \gg E(p_{\rm Fermi})$
"lifts" the degeneracy → recover classical ideal gas
this will be explosively crucial for the fate of Sun-like stars!

∞

Poll: Nuclei/Ions

Vote your conscience! *All answers get credit*

So far we have focused on degenerate electron gas
What about the nuclei they came with?
What sets the relationship between nuclei and e?

A must balance energy: energy densities must be equal

B must balance momentum: Fermi momenta must be equal

C must balance pressure: pressures must be equal

D must balance charge: p and e numbers must be equal

Mass Density of a Degenerate Electron Gas

electron number density: $n_e = 8\pi/3h^3 \; p_{\rm Fermi}^3$

electron mass density

$$\rho_e = m_e n_e = \frac{8\pi}{3} \frac{m_e p_{\rm Fermi}^3}{h^3} \tag{9}$$

but there are nuclei (positive ions), giving net charge zero!
so total p and e densities must balance: $n_{p,\rm tot} = n_e$

if average ion charge is Z_i and mass $m_i = A_i m_p$:
total proton number density $n_p = Z_i n_i$
ion number density $n_i = n_e/Z_i$
ion mass density $\rho_i = m_i n_i = A_i m_p n_e / Z_i$

total mass density $\rho = \rho_e + \rho_i \approx \rho_i$: dominated by ions

Pressure of a Degenerate Electron Gas

in studying ideal gas, found that pressure
is an average momentum flow:

$$P = \text{momentum per particle} \times \text{particle flux} = \frac{1}{3}\langle p \; v \; n \rangle \qquad (10)$$

where $v(p)$ is the velocity for momentum p
if *non-relativistic:* $v = p/m$

for degenerate electron gas, pressure is

$$P_e = \frac{1}{3}\int p \; v \; dn_e = \frac{8\pi}{3h^3}\int_0^{p_{\rm Fermi}} p \; v \; p^2 \; dp \qquad (11)$$

$$= \frac{8\pi}{15 m_e h^3} \, p_{\rm Fermi}^5 \qquad (12)$$

but Fermi momentum given by number density: $p_{\rm Fermi} \sim n_e^{1/3} h$

$$P_e = \frac{8\pi h^2}{15 m_e}\left(\frac{3n_e}{8\pi}\right)^{5/3} \qquad (13)$$

Non-Relativistic Degeneracy Pressure

for (cold) non-relativistic degenerate electrons

$$P_{e,\mathrm{nr}} = \frac{8\pi h^2}{15 m_e} \left(\frac{3 n_e}{8\pi} \right)^{5/3} \tag{14}$$

- pressure only depends on density and not temperature
- pressure grows with density

  $$P_{e,\mathrm{nr}} \propto n_e^{5/3} \propto \rho^{5/3}$$

- degeneracy pressure is large even when temperature small!
  due to Pauli principle! a quantum effect!
  contrast classical ideal gas: $P = nkT \to 0$ as $T \to 0$
- sometimes useful to write $P_e = K_{\mathrm{nr}}\ n_e^{5/3}$, with

$$K_{\mathrm{nr}} = \left(\frac{3}{8\pi} \right)^{2/3} \frac{h^2}{5 m_e} \tag{15}$$

A Non-Relativistic Degenerate Star

consider a star of mass M and radius R
made of a non-relativistic degenerate gas
so pressure is $P = K_{\rm nr}\ n_e^{5/3}$

equate this to the central pressure $P_c \sim GM^2/R^4$:

$$K_{\rm nr} \left(\frac{M}{R^3}\right)^{5/3} \sim \frac{GM^2}{R^4} \tag{16}$$

$$K_{\rm nr} \frac{M^{5/3}}{R^5} \sim \frac{GM^2}{R^4} \tag{17}$$

so the stellar radius:

$$R \sim \frac{K_{\rm nr}}{G}\ M^{-1/3} \tag{18}$$

and for $M = 1 M_{\odot}$ with 2 nucleons per electron, estimate

$$R_{\rm degen}(1 M_{\odot}) \sim\ 10^4\ \text{km} \sim 2\, \text{R}_{\rm Earth} \tag{19}$$

Q: what does this imply? are there objects like this?

White Dwarfs: Degenerate Stars

we see that a degenerate star is *incredibly compact!*

$$R_{\text{degen,nr}} \sim \frac{K_{\text{nr,degen}}}{G} \frac{1}{M^{1/3}} \quad (20)$$

$$R_{\text{degen,nr}}(1 M_{\odot}) \sim 10^4 \text{km} \sim 2 \text{R}_{\text{Earth}} \quad (21)$$

mass of the Sun packed into Earth-sized volume
as expected for a maximally dense object

compared to stars we know, radius is:
- *tiny* compared to the Sun, giants, and supergiants
- but is exactly in line with white dwarfs

white dwarfs are degenerate stars!
- supported by degeneracy pressure
- somehow resulting from incredible compression which left high temperature (hence white)

White Dwarfs Observed

nearest white dwarf is **Sirius B**: unseen by naked eye
but companion of Sirius A, brightest star in sky

binary system, so mass known: $M(\text{Sirius B}) = 1.02 M_{\odot}$
but radius about Earth-sized! (PS7)

`www: Sirius B in optical`
`www: Sirius B in X-ray` – outshines Sirius A!
Q: what does this mean?

mass-radius relation:

$$R_{\text{degen,nr}} \sim \frac{K_{\text{nr,degen}}}{G} \frac{1}{M^{1/3}} \tag{22}$$

Q: radius if more massive? less? how to test?

White Dwarfs Radius and Mass

white dwarfs: $R_{\text{nr,degen}} \sim M^{-1/3}$
so larger mass means smaller radius!
white dwarfs get more compact when adding mas!

to test: compare radii for white dwarfs with different masses

40 Eridani B: Trekkers–this is Vulcan's system, with a confirmed planet!

$$M(\text{40 Eri B}) = 0.50 M_\odot \approx M(\text{Sirius B})/2 \quad (23)$$
$$R(\text{40 Eri B}) = 1.7 R(\text{Sirius B}) \quad (24)$$

indeed smaller mass $\rightarrow$ larger radius

Q: how does average density depend on mass for a white dwarf?
Q: what if we keep adding mass to a white dwarf?

White Dwarfs: Increasing Mass

white dwarfs: $R_{\text{nr,degen}} \sim M^{-1/3}$
so average density grows with mass!

$$\rho_{\text{nr,degen}} \sim \frac{M}{R^3} \;\propto\; M^2 \tag{25}$$

adding mass $\rightarrow$ smaller size, higher density
eventually density so high: Fermi level $p_0 \sim n_e^{1/3} h \gg m_e c$
star becomes relativistic degenerate!

to understand these objects
must understand the relativistic degenerate case