Astro 404
Lecture 24
Oct. 20, 2021

Announcements:

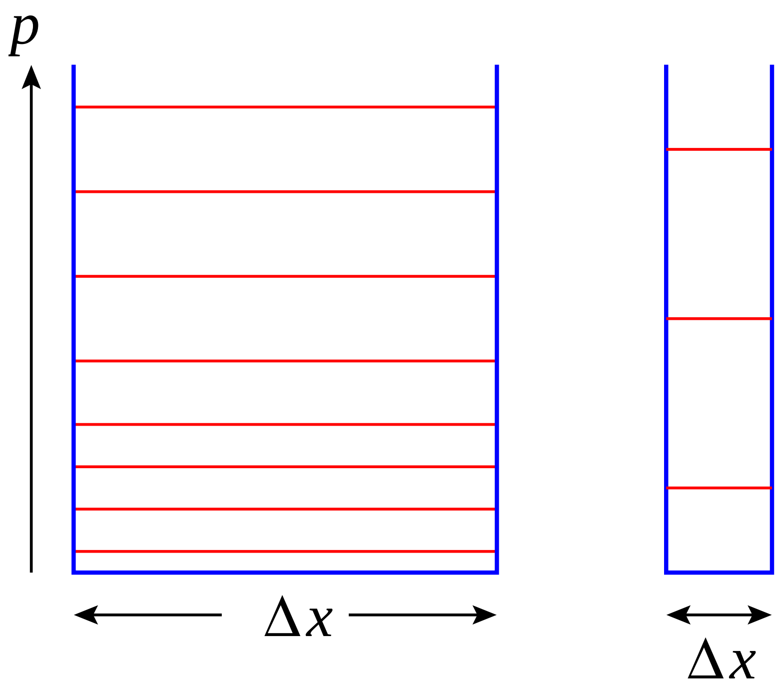
- **PS7 due Friday**
- Office Hours:
  Instructor: Wed 11–noon, or by appointment
  TA: Thursday 2:30–3:30
- Exam: grading elves hard at work

Last time: degeneracy and stars

- quantum uncertainty principle $\Delta x \, \Delta p > h$
  the more particles confined (smaller Δx)
  the larger the spread of momenta (Δp) and energy
  even if temperature is small or zero!
- Pauli principle: only one quantum state per electron
  so large number of electrons $\rightarrow$ high momentum and energy

Quantum Gas and Particle Density

$$\Delta x \;\Delta p \;> h$$



- Denser $\rightarrow$ smaller Δx $\rightarrow$ higher momentum states

When is a Gas Degenerate?

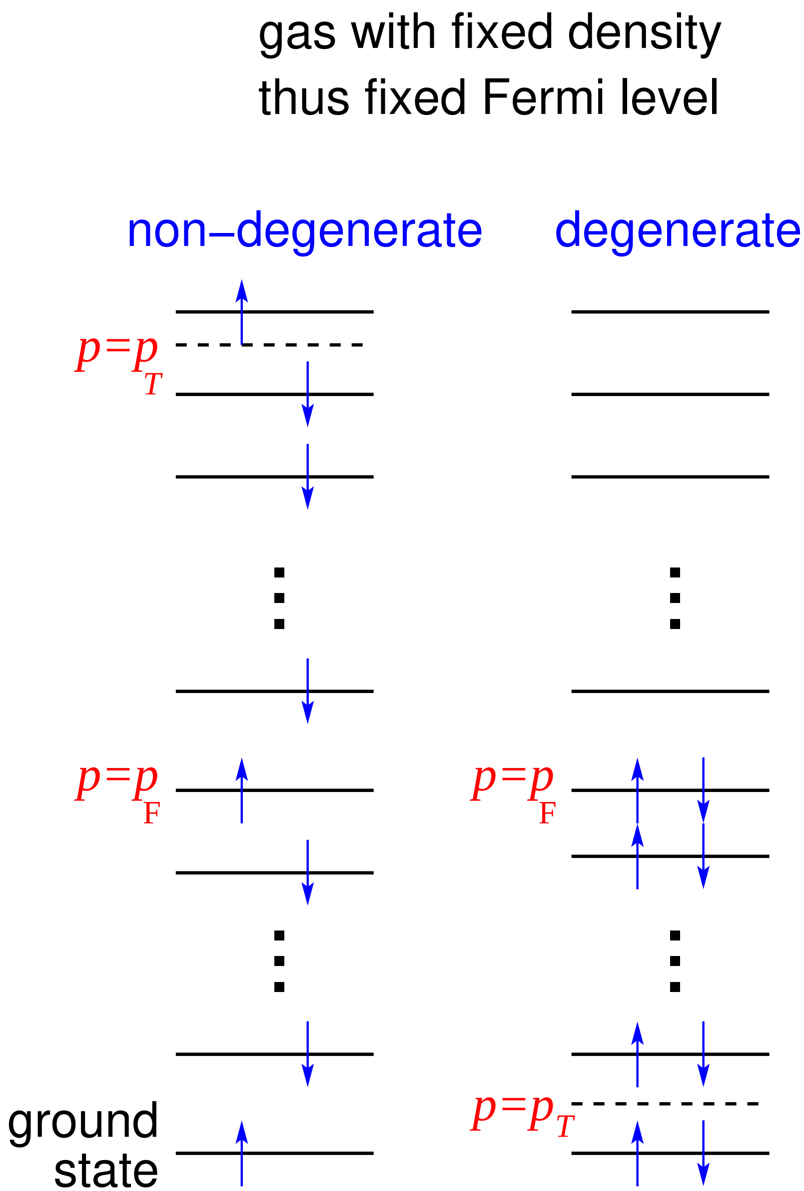
complete degeneracy: $T \to 0$

- gas particles fill quantum levels fully, from ground up
- no vacant levels until the highest
- highest level = Fermi level, momentum p_{F} and energy E_{F}
  number density $n = 4\pi g p_{\mathsf{F}}^3 / 3h^3$
- lowest energy configuration for density n
- particle energies are due to quantum motion
  average particle energy $< E_{\mathsf{F}}$

Now raise $T > 0$

additional thermal energy available

Q: What happens to level populations if $kT \ll E_{\mathsf{F}}$? if $kT \gg E_{\mathsf{F}}$?



non-degenerate:

- quantum levels sparsely filled
- average particle energy $kT \gg E_\mathsf{F}$
  or equivalently $p_T \gg p_\mathsf{F}$
- particle energies dominated
  by thermal motion
- **behaves as an ideal gas**

degenerate:

- quantum levels fully filled
  up to Fermi level
- average particle energy $kT \ll E_\mathsf{F}$

Q: what's the procedure to find if a gas is degenerate?

Degeneracy Detection Procedure: Electron Gas

- Find electron number density

$$n_e = \frac{\rho}{\mu_e m_p} \tag{1}$$

- this sets Fermi momentum via

$$n_e = \frac{8\pi}{3}\left(\frac{p_\mathrm{F}}{h}\right)^3 \tag{2}$$

- From p_F find corresponding Fermi energy E_F
  - ▷ if $p_\mathrm{F} < m_e c$: non-relativistic, $E_\mathrm{F} = p_\mathrm{F}^2/2m_e$
  - ▷ if $p_\mathrm{F} > m_e c$: relativistic, $E_\mathrm{F} = c p_\mathrm{F}$

- compare kT and E_F:

degenerate if $E_\mathrm{F} \gg kT$
non-degenerate if $E_\mathrm{F} \ll kT$

Pressure: Overview

pressure reflects motion of particles in gas
in fact: P measures momentum flow
in general: depends on density and temperature
but details depend on the particles (see PS7 for more)

- how compact relative to de Broglie wavelength:
  non-degenerate vs degenerate
- how fast compared to c
  non-relativistic vs relativistic

non-degenerate:
- non-relativistic: ideal gas $P = nkT = \rho kT/m_{\text{g}}$
- relativistic: $P_\gamma = aT^4/3$

degenerate:
- non-relativistic gas: $P \propto \rho^{5/3}$
- relativistic: ???

White Dwarfs: Size and Density

white dwarfs: degenerate stars!
supported by degeneracy pressure

hydrostatic equilibrium: $P_{\text{degen,nr}} = GM^2/R^4$ gives
radius $R_{\text{nr,degen}} \sim M^{-1/3}$

$$\rho_{\text{nr,degen}} \sim \frac{M}{R^3} \propto M^2 \tag{3}$$

adding mass $\rightarrow$ smaller size, higher density

How does this work?
Q: why does more mass force the star to be smaller?

White Dwarf: Response to Mass

White dwarfs supported by degeneracy pressure

a more massive WD:

- stronger gravity
- requires larger pressure support
  and $P_{\text{degen,nr}} \propto \rho^{5/3}$ depends only on density
  not on temperature – so only option is ...
- star becomes smaller so that density ρ increases

eventually density so high: Fermi level $p_{\text{f}} \sim n_e^{1/3} h \gg m_e c$
star becomes relativistic degenerate!

∞ to understand these objects
must understand the relativistic degenerate case

What Makes a Particle Relativistic?

individually and in groups

particles behave differently if they are relativistic vs not but how do we know when these apply?

Newtonian physics: developed for slow particles: $v \ll c$

these are non-relativistic, and familiar

for particle of mass m

- non-relativistic momentum $p_{\mathrm{nr}} = mv$
- non-relativistic kinetic energy $E_{\mathrm{k,nr}} = mv^2/2 = p^2/2m$

Special Relativity

Einstein: Newtonian physics fails when speeds $v \to c$
need to rethink particle dynamics!

- **relativistic momentum**

$$p_{\text{rel}} = \frac{mv}{\sqrt{1 - v^2/c^2}}$$

- **relativistic total energy**

$$E_{\text{rel}} = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \sqrt{(cp_{\text{rel}})^2 + (mc^2)^2}$$

Q: Einstein results for $v = 0$? for $v \ll c$?

relativistic momentum $p_{\text{rel}} = mv/\sqrt{1 - v^2/c^2}$

relativistic energy $E_{\text{rel}} = mc^2/\sqrt{1 - v^2/c^2}$

for $v = 0$:

$p_{\text{rel}} = 0$: agrees with Newtonian result for $v = 0$

$E_{\text{rel}} = mc^2$: even with no motion, particle mass contains energy!

"**rest mass energy**" or "rest energy" is $E_{\text{rest}} = mc^2$

for $v \ll c$: PS7 shows that

$p_{\text{rel}} \to mv$ – recover Newtonian result!

$$E_{\text{rel}} = mc^2 \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) = mc^2 + \frac{1}{2}mv^2 + \cdots \approx E_{\text{rest}} + E_{\text{k,nr}}$$

energy just due to motion is Newtonian kinetic energy!

iClicker Poll: When are We Relativistic?

Which of these indicates a particle is relativistic?
that is, need to use special relativity momentum, energy?

A $v \sim c$

B $p \gtrsim mc$

C $E_\mathsf{k} \gtrsim mc^2$

D (a), (b), and (c) are all true if any are true

PS7: show that $v \ll c$ also means $p \ll mc$ and $E_\mathrm{k} \ll mc^2$
this is the non-relativistic limit
opposite is ultra-relativistic limit

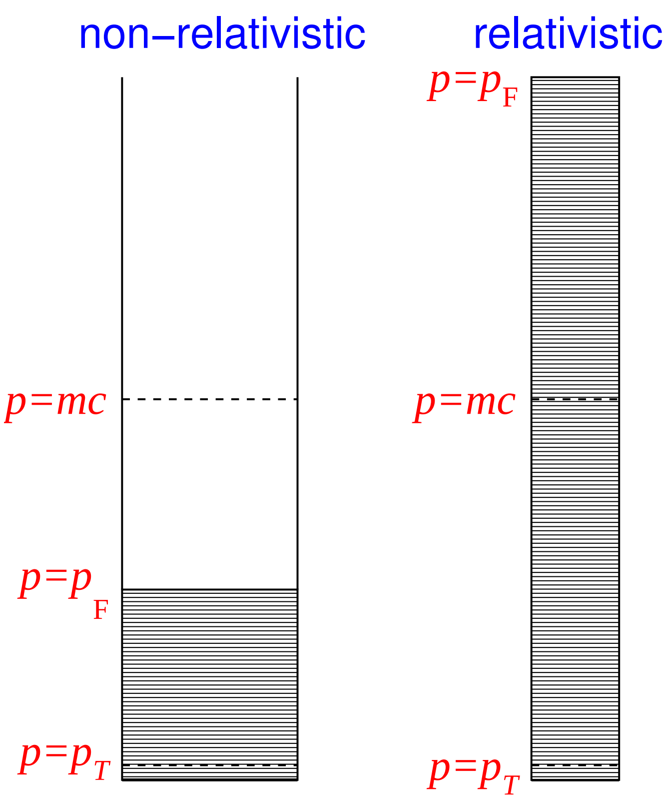
ultra-relativistic limit: $v \to c$:

- $p \to E/c$
- $E \gg mc^2$

for photons (massless particles):
use $E = \sqrt{(cp)^2 + (mc^2)^2} = cp$
same as ultra-relativistic limit for massive particle!

Q: so when is a degenerate gas relativistic?

Cold Degenerate Gases



relativistic or not?

compare p_F to mc

- $p_F \ll mc$: non-rel
- $p_F \gg mc$1: relativistic

iClicker Poll: Relativistic Pressure

Vote your conscience! *All answers get credit*

Non-relativistic degenerate gases have $P_{\text{nr}} \propto \rho^{5/3}$

What will we find for relativistic degenerate gases?

A also $P_{\text{rel}} \propto \rho^{5/3}$

B P_{rel} increases more strongly with ρ

C P_{rel} increases less strongly with ρ

Now Go Relativistic

if electrons are relativistic: speeds $v \approx c$, momenta $p \gg m_e c$
and energy $E = \sqrt{(m_e c^2)^2 + (cp)^2} \to cp$

but free particle states still set by de Broglie wavelength
and still labeled by momentum $\lambda_{\rm deB} = h/p$

number density still $n_e = 8\pi/3 \ p_{\rm Fermi}^3/h^3$
pressure uses $v(p) \approx c$:

$$P_{e,\rm rel} = \frac{1}{3}\langle p \ v \ n\rangle = \frac{2\pi}{3}\frac{cp_{\rm Fermi}^4}{h^3} \quad (4)$$

$$= \frac{2\pi}{3}hc\left(\frac{3n_e}{8\pi}\right)^{4/3} \quad (5)$$

- relativistic pressure also only depends on density and not T
- pressure grows with density $P_{e,\rm rel} \propto n_e^{4/3} \propto \rho^{4/3}$
- weaker power law than non-relativistic scaling
- $P_{\rm rel} = K_{\rm rel} \ n_e^{4/3}$ with $K_{\rm rel} = (3/8\pi)^{1/3} hc/4$

Relativistic Degenerate Stars: Theory

non-relativistic degenerate stars : $R_{\text{nr,degen}} \sim M^{-1/3}$
so average density grows with mass!

$$\rho_{\text{nr,degen}} \sim \frac{M}{R^3} \propto M^2 \tag{6}$$

adding mass $\rightarrow$ smaller size, higher density
eventually density so high: Fermi level $p_0 \sim n_e^{1/3} h \gg m_e c$
star becomes relativistic degenerate!

for relativistic degenerate gas: $P_{\text{rel,degen}} = K_{\text{rel,degen}} \rho^{4/3}$
and equating this to $P_c \sim GM^2/R^4$ gives

$$K_{\text{rel,degen}} \left(\frac{M}{R^3}\right)^{4/3} \sim \frac{GM^2}{R^4} \tag{7}$$

$$K_{\text{rel,degen}} \frac{M^{4/3}}{R^4} \sim \frac{GM^2}{R^4} \tag{8}$$

$$M^{2/3} \sim \frac{G}{K_{\text{rel,degen}}} \tag{9}$$

Relativistic Degenerate Stars: Predictions

hydrostatic equilibrium in degenerate relativistic star gives mass

$$M \sim \left(\frac{G}{K_{\text{rel,degen}}}\right)^{3/2} \quad (10)$$

- radius drops out!
- also independent of density
- only a single, unique mass works!

numerically:

$$\boxed{M = M_{\text{Chandra}} = 1.4 M_{\odot}}$$

Chandrasekhar limit! S. Chandrasekhar 1931(!!)

Q: what if white dwarf has $M < M_{\text{Chandra}}$*?*
Q: what if white dwarf has $M > M_{\text{Chandra}}$*?*

if high-density WD has $M < M_{\text{Chandra}}$
then pressure (more than) enough to balance gravity
$\rightarrow$ *WD is stable against collapse*

but: *if high-density WD has $M > M_{\text{Chandra}}$*
then pressure *not enough* to balance gravity
$\rightarrow$ gravity force not balanced
$\rightarrow$ *star unstable $\rightarrow$ collapses under its own weight!*
$\rightarrow$ catastrophe!

prediction: *Chandrasekhar mass is*
maximum *mass of white dwarfs!*
i.e., most massive possible "ordinary solid" = supported by e degeneracy

`www: Chandrasekhar (1931) paper` – you can understand this!
Q: how to test this prediction with white dwarf data?

iClicker Poll: White Dwarf Masses

Vote your conscience! *All answers get credit*

white dwarf masses are observed in hundreds of binary systems
What do we find?

A *most* masses $M_{\rm wd} < 1 M_\odot$, none $> M_{\rm Chandra}$

B *most* masses $1 M_\odot < M_{\rm wd} < M_{\rm Chandra}$, none $> M_{\rm Chandra}$

C roughly equal masses up to $M_{\rm Chandra}$

D about 10% of white dwarfs have $M_{\rm w} > M_{\rm Chandra}$

White Dwarf Masses Observed

observed white dwarf masses

- most found at $M_{\mathrm{wd}} \lesssim 1 M_\odot$

but there is an *observational bias*

- white dwarf cooling sequence: $L_{\mathrm{wd}} = 4\pi R^2 \sigma T_{\mathrm{eff}}^4 \propto M^{-2/3} T_{\mathrm{eff}}^4$
  less massive are more luminous – easier to find

but even after accounting for this bias

- true distribution dominated by $< 1 M_\odot$ white dwarfs
- *no white dwarfs found with* $M > 1.4 M_\odot$

implications:

- Chandrasekhar mass truly limits white dwarf masses
- lack of high mass WDs suggests *either*
  these are rarely made, or these don't survive when made

White Dwarf Cooling

White dwarfs are hot–hence the name
but supported by degeneracy pressure, not thermal pressure!
$p_T \ll p_{\mathrm{F}}$
not heated from nuclear reactions

no energy source $\rightarrow$ white dwarfs cool over time
$\Rightarrow$ a white dwarf *is* like a cup of coffee!

luminosity gives energy loss: $L = 4\pi R^2 \sigma T_{\mathrm{eff}}^4$
but for non-relativistic white dwarfs $R \propto M^{-1/3}$, so

$$L_{\mathrm{wd}} \propto M^{-2/3} T_{\mathrm{eff}}^4$$

- at fixed T_{eff} more massive $\rightarrow$ *less luminous!*
- degeneracy pressure nearly independent of temperature
  so cooling doesn't change R, only T_{eff}
  Q: so for a fixed WD mass, how does L_{wd} change?

White dwarf luminosity:

$$L_{\rm wd} \propto M^{-2/3} T_{\rm eff}^4$$

for fixed mass cooling drops $T_{\rm eff}$
and thus drops $L_{\rm wd} \propto T_{\rm eff}^4$

so on a plot of $y = \log L$ vs $x = \log T_{\rm eff}$
we expect white dwarfs of a fixed mass to fall on line:
"white dwarf cooling sequence"

$$y = \log L \;=\; 4 \log T_{\rm eff} - \frac{2}{3} \log M + {\rm const} \qquad (11)$$

$$\;=\; 4x - \frac{2}{3} \log M + {\rm const} \qquad (12)$$

so test by looking at HR diagram
Q: expectations if all WD have same mass? if a range of masses?
`www: Gaia white dwarfs`