Astro 404

Lecture 25

Oct. 22, 2021

Announcements:

- **PS7 due today at 5pm**
- **PS8 due next Friday**

Last time: degenerate stars – white dwarfs

- hydrostatic support from degeneracy pressure
- non-relativistic degenerate stars *Q:* $P(\rho)$*?*
  Q: size R *vs mass* M*? implications?*

Recap: Degenerate Stars–White Dwarfs

for a non-relativistic degenerate gas: $P_{\mathrm{degen,NR}} = K_1 \rho^{5/3}$
a star supported by NR degeneracy pressure has

$$R \propto \frac{1}{M^{1/3}} \qquad \rho \sim \frac{M}{R^3} \propto M^2 \tag{1}$$

higher mass $\rightarrow$ smaller! denser! opposite of Main Seq trend
these are white dwarfs! mass/radius relation observed!

but when density becomes high, $p_{\mathrm{F}} \propto \rho^{1/3} > m_e c$

- degenerate gas becomes relativistic
- then hydrostatic equilibrium allows only one mass
  Chandrasekhar mass – maximum mass of a white dwarf
  $\boxed{M = M_{\mathrm{Chandra}} = 1.4 M_\odot}$ higher mass: hydrostatic equilibrium no longer possible! yikes!

Polytropes

We see that cold degenerate gas pressure

- that depends only on density
- and varies as a power law: $P \propto \rho^{\gamma}$

generalize to *polytropic equation of state*

$$P = K\ \rho^{\gamma} = K\ \rho^{1+1/n} \tag{2}$$

- pressure depends only on density, with
- K a constant
- $n = 1/(\gamma - 1)$ called the polytrope index

no temperature dependence: simplifies modeling
mechanical structure decoupled from thermal structure

Hydrostatic Equilibrium of a Polytrope

use polytropic equation of state

$$P = P(\rho) = K\ \rho^{\gamma} = K\ \rho^{1+1/n} \tag{3}$$

hydrostatic equilibrium gives

$$\frac{dP}{dr} = -\frac{Gm(r)\ \rho(r)}{r^2} \tag{4}$$

isolate enclosed mass term

$$\frac{r^2}{\rho}\frac{dP}{dr} = -Gm(r) \tag{5}$$

now recall enclosed mass $dm = 4\pi r^2\ \rho\ dr$ so

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G r^2 \rho \tag{6}$$

hydrostatic equilibrium, rewritten

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G r^2 \rho \tag{7}$$

for polytrope $P = K\rho^\gamma$, can write this as

$$\frac{K}{4\pi G}\ \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d\rho^\gamma}{dr}\right) = -\rho \tag{8}$$

a version of the **Lane-Emden equation**

a differential equation that only depends on ρ

• so solution gives $\rho(r)$

• which also gives $P(r) = K\ \rho(r)^\gamma$

$\Rightarrow$ determines structure of star!

PS8: solve Lane-Emden for a simple-ish case

Polytropes, Degenerate Matter, and White Dwarfs

polytrope: matter where pressure depends only on density
that is, equation of state has form $P(\rho)$ and not $P(\rho, T)$
degenerate matter is example of this
usually polytropes are approximately power laws: $P = K\rho^{\gamma}$

in this case, hydrostatic equilibrium alone determines $\rho(r)$
that is: hydrostatic force balance alone sets mass distribution
independent of interior temperature, indeed for $T = 0$, so:
mechanical structure $\rho(r)$ independent of thermal structure $T(r)$

non-relativistic degenerate electron matter:

$$P_{e,\mathrm{nr}} = K_{e,\mathrm{nr}} \; n_{\mathrm{e}}^{5/3} \; = K_{\mathrm{nr,degen}} \; \rho^{5/3}$$

gives $R \propto M^{-1/3}$: higher mass $\leftrightarrow$ smaller white dwarf!

Dynamical Stability of Stars

Why do relativistic degenerate stars become *unstable*
while non-relativistic degenerate stars do not?

look at *response to perturbations*

consider a star, mass M and radius R

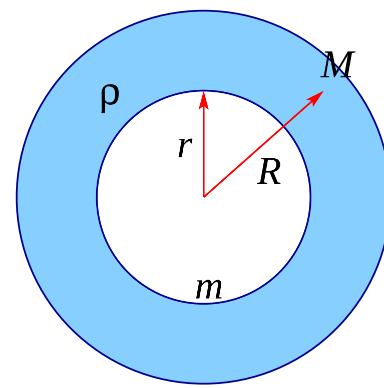
in *hydrostatic equilibrium*

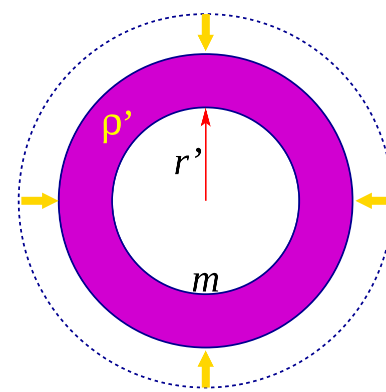
mass shell m at radius $r(m)$ feels weight per area

$$\frac{F_{\text{grav}}}{A} = G \int_m^M \frac{m\ dm}{4\pi r^4}$$

where $dm = 4\pi r^2\ \rho\ dr$ or

$$\rho = \frac{1}{4\pi r^2} \frac{dm}{dr}$$



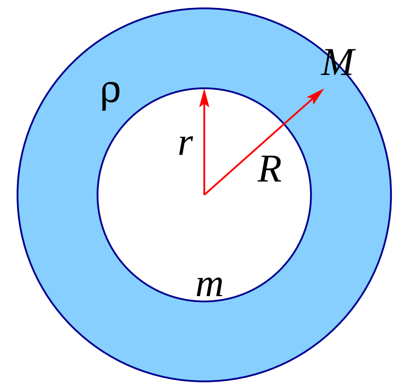


compress shell a small amount:

$r' = r - \epsilon r = (1 - \epsilon) r$

Q: response of gas P_{gas}*?* F_{grav}/A

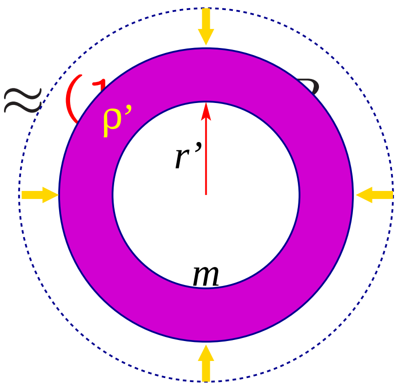
compress shell a small amount $r' = r - \epsilon r = (1-\epsilon)$
while hold shell enclosed mass m fixed



gravitational response: weight per area

$$(F_{\rm grav}/A)' = G\int_m^M \frac{m\ dm}{4\pi(1-\epsilon)^4 r^4} = (1-\epsilon)^{-4}\left(\frac{F_{\rm grav}}{A}\right) \approx (1$$

where we used $(1-\epsilon)^s \approx 1 - s\epsilon + \cdots$



gas pressure response:

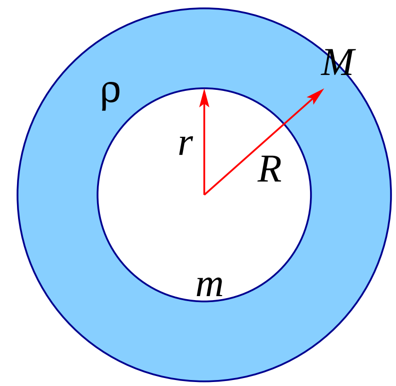
$$\rho' = \frac{1}{4\pi(1-\epsilon)^2 r^2}\frac{dm}{dr'} = (1-\epsilon)^{-2}\frac{1}{4\pi r^2}\frac{dm}{dr}\frac{dr}{dr'} = \frac{\rho}{(1-\epsilon)^3} \approx (1+3\epsilon)\rho$$

∞ for a polytrope: $P = K\rho^\gamma$, so

$$P'_{\rm gas} = K(\rho')^\gamma \approx (1-\epsilon)^{3\gamma}P \approx (1+3\gamma\epsilon)P \qquad (9)$$

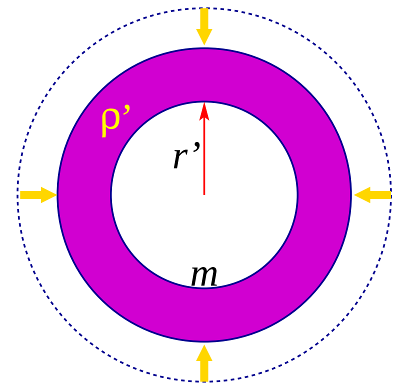
so after perturbation $r' = (1 - \epsilon) r$

new pressures are in general no longer the same



so look at pressure difference

$$P'_{\rm gas} - (F_{\rm grav}/A)' \approx (1 + 3\gamma\epsilon)P - (1 + 4\epsilon)P$$
$$= \left(\gamma - \frac{4}{3}\right) 3\epsilon P$$



Q: response if $\epsilon = 0$*?*

Q: response if $\gamma > 4/3$*?*

Q: response if $\gamma < 4/3$*?*

Q: implications?

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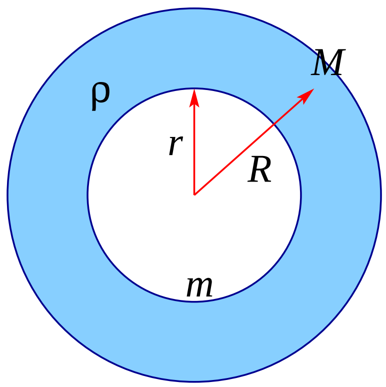

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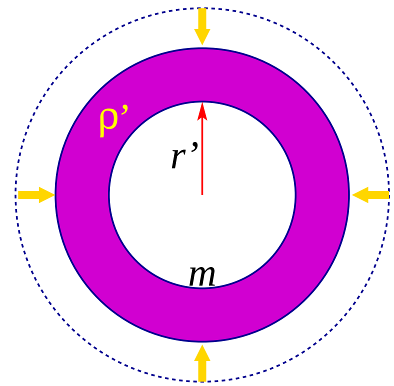


Q: response if $\epsilon = 0$*?*

Q: response if $\gamma > 4/3$*?*

Q: response if $\gamma < 4/3$*?*

Q: implications?

Polytropic Index and Dynamical Stability

after *radial compression* $r'(m) = (1-\epsilon) r(m)$:

$$P'_{\text{gas}} - (F_{\text{grav}}/A)' \approx (1+3\gamma\epsilon)P - (1+4\epsilon)P = \left(\gamma - \frac{4}{3}\right) 3\epsilon P \quad (10)$$

$\boxed{\text{if } \gamma > 4/3}$

outward gas pressure grows faster

than inward weight per area due to gravity of outer layers

net outward pressure: gas expands back to original size

restoring force opposes perturbation $\rightarrow$ *stable equilibrium*

$\boxed{\text{if } \gamma < 4/3}$

net inward pressure: restoring force enhances perturbation

runaway $\rightarrow$ equilibrium not restored $\rightarrow$ *dynamically unstable!*

if $\gamma = 4/3$

no restoring force at all!

but this means *perturbation not undone*

lesson: instability if $\gamma \leq 4/3$!

so for white dwarfs: degenerate stars

- low mass $\leftrightarrow$ Fermi momentum $p_0 < m_e c$
  degenerate electrons are non-relativistic
  $P \propto \rho^{5/3}$: $\gamma = 5/3$ means stability!
- but as mass increases, Fermi momentum p_0 increases
  to $p_0 \geq m_e c$: electrons become relativistic
- as $M \to M_{\text{Chandra}}$, then $\gamma \to 4/3$
  fully relativistic $\to$ unstable!

This has grave consequences for stellar evolution

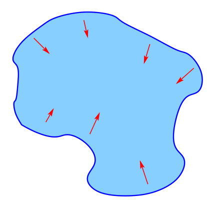
Star Formation: Birth to Main Sequence

Star Formation

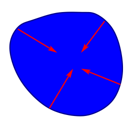
the formation of stars (and planets) remains mysterious
deserves its own course: **Astronomy 405**
offered this coming semester!

here: we sketch some highlights
so you see how (proto)stars approach the main sequence

basic idea:



- huge, *low-density interstellar gas clouds*
- collapse under their own gravity



- and likely fragment into “protostellar cores”
- which contract until nuclear reactions ignite to signal marking the *zero age main sequence*

iClicker Poll: Fuel for Star Formation

interstellar gas clouds exist in several forms

Which of these is most favorable to gravitational collapse?

Hint: we now want to be *out* of hydrostatic equilibrium!

A ionized gas: mostly free p and e

B atomic gas: mostly $\mathrm{H} = \boxed{\mathrm{pe}}$ atoms

C molecular gas: mostly $\mathrm{H_2} = \boxed{\mathrm{HH}}$ molecules

Star Formation: Raw Material

to collapse, clouds must *not* be in hydrostatic equilibrium!
gravity must overwhelm pressure gradients

at low interstellar densities: classical ideal gas (non-degenerate)
$P = n\ kT$: low T means low P

but kT also sets particle kinetic energy scales
compare to **binding energy**
B = energy to tear gas particle apart

- molecular hydrogen: $B(\mathrm{H_2}) = 4.5$ eV
- atomic hydrogen: $B(\mathrm{H}) = 13.6$ eV
- ionized hydrogen: already torn apart–unbound!

Q: lessons?

Molecular Gas is Star Formation Fuel

lessons:

- molecular hydrogen has smallest binding energy
  requires coldest temperatures to survive collisions
- as T rises, molecules $\rightarrow$ torn to atoms $\rightarrow$ torn to ions
- *collapse and star formation most likely in* molecular gas

our Galaxy and other galaxies contain **giant molecular clouds**

- made mostly of molecular hydrogen H_2
- but most easily seen via CO carbon monoxide molecules
- typical giant molecular cloud conditions
- mass $M \sim 10^5 M_\odot$, size $R \sim 10$ pc, temperature $T \sim 20$ K
  can be opaque to optical light, visible in IR and radio

`www: molecular clouds`