Astronomy 501: Radiative Processes

Lecture 10

Sept 14, 2022

Announcements:

- **Problem Set 3 due Friday**
- Office Hours: BDF–after class or by appointment
  Chris: 11:30-12:30 tomorrow

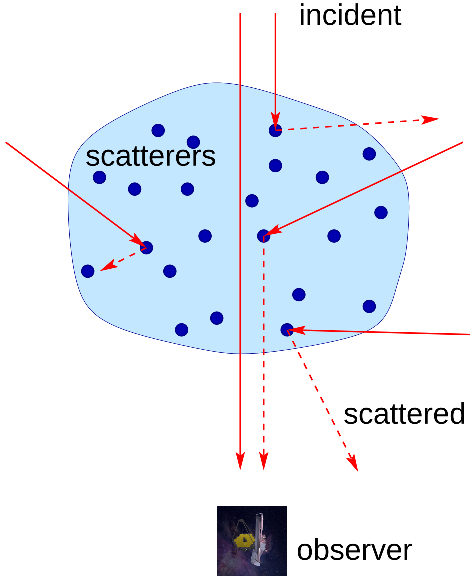
Last time: radiation transfer for lines

Today: scattering

Scattering

Pure Scattering

Consider an idealized case with radiation propagating through a medium with "*pure scattering*," i.e., scattering, but *no emission*, and *no absorption*

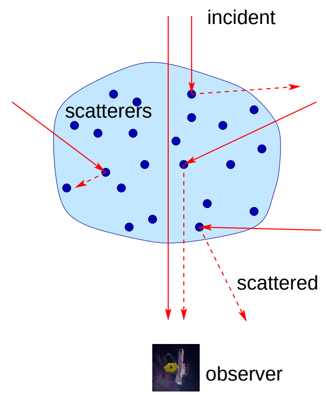


Q: what are effects of pure scattering on incident radiation field?

Q: is anything preserved?

Effects of Pure Scattering

scattering preserves/conserves photon *number*
but *redistributes* photon direction

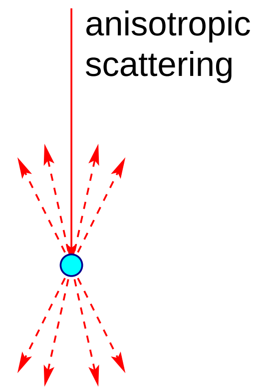


Recall: intensity in a ray is a *directional* quantity
i.e., really $I_\nu = I_\nu(\theta, \phi) = I_\nu(\hat{n})$,
with $\hat{n}$ a unit vector toward (θ, ϕ)

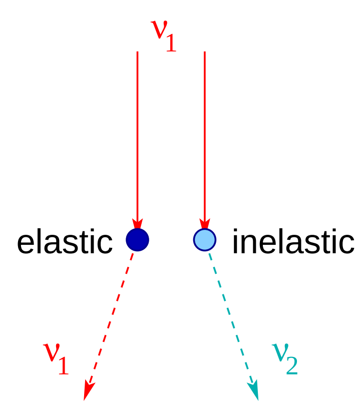
generally: to find scattering into *one direction* (one ray)
must sum contributions from intensity in *all other directions!*
...and at every point in space!

Scattering Anisotropy and Inelasticity

in general:
different probability to scatter
into different angles: *anisotropic*

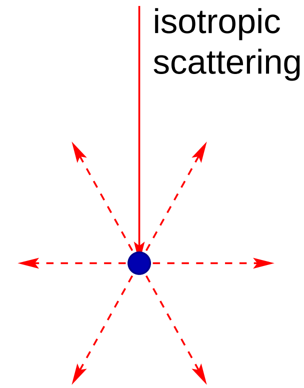


in general:
photon energy and changes
exchanged with scatterer
and thus change in ν, λ
"inelastic" scattering



scattering is in general very difficult to calculate
in practice: evaluate numerically (simulate!)

Special Case: Coherent Isotropic Scattering



consider a special case:

- *isotropic* scattering:
  no preferred direction for scattered photons

- photon energy unchanged ("*coherent or elastic scattering*")
  good approximation for scattering by non-relativistic e

define **scattering coefficient** $\varsigma_\nu = n_{\rm scat} \sigma_{{\rm scat},\nu}$,
and thus also scattering cross section $\sigma_{\rm scat}$, such that
intensity lost to scattering *out* of ray is

$$dI_\nu = -\varsigma_\nu \; I_\nu \; ds \tag{1}$$

isotropic scattering $\rightarrow \varsigma_\nu$ same for all directions

Q: what is intensity scattered into *the ray?*

Isotropic Coherent Scattering

intensity scattered *out* of ray $I_\nu(\hat{n})$ is

$$dI_\nu(\hat{n}) = -\varsigma_\nu \; I_\nu(\hat{n}) \; ds \tag{2}$$

if scattering *isotropic*, the portion *into* $\hat{n}$ *from* $\widehat{n'}$ is

$$dI_\nu(\hat{n}) = \frac{d\Omega'}{4\pi} \left| dI_\nu(\hat{n}') \right| \tag{3}$$

and so integrating over all possible solid $d\Omega'$ gives

$$dI_\nu(\hat{n}) = \frac{\varsigma_\nu}{4\pi} \int I_\nu \; d\Omega \; ds = \varsigma_\nu \; J_\nu \; ds \tag{4}$$

where J_ν is the angle-averaged intensity and thus for isotropic coherent scattering

$$\frac{dI_\nu(\hat{n})}{ds} = -\varsigma_\nu \left[I_\nu(\hat{n}) - J_\nu\right] \tag{5}$$

and so the source function is

$$S_\nu = J_\nu \tag{6}$$

and the transfer equation can be written

$$\frac{dI_\nu(\hat{n})}{d\tau_\nu} = -I_\nu(\hat{n}) + J_\nu \tag{7}$$

where mean flux $J_\nu = \int I_\nu(\hat{n}')d\Omega'/4\pi$, and $d\tau_\nu = \varsigma_\nu \; ds$

Q: why is this intuitively correct?

Q: what is effect on I_ν of many scattering events?

for coherent, isotropic scattering:

$$\frac{dI_\nu(\hat{n})}{d\tau_\nu} = -I_\nu(\hat{n}) + J_\nu \tag{8}$$

depends on I_ν field in *all directions*

$\Rightarrow$ scattering couples intensity in different directions

if many scattering events, τ_ν large: $I_\nu \to J_\nu$

after large number of mean free paths, photons $\to$ isotropic

$\Rightarrow$ (isotropic) scattering randomizes photon directions

reduces anisotropy

transfer with scattering: integro-differential equation

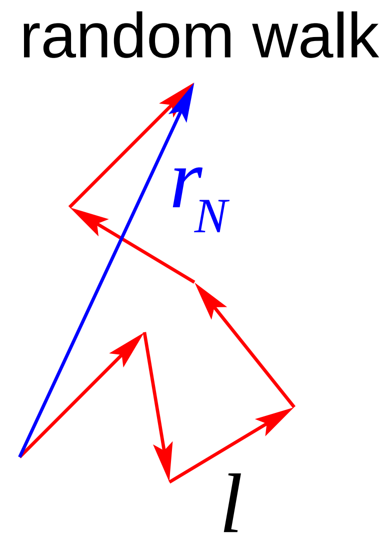
generally very hard to solve!

Q: transfer equation modification for anisotropic scattering?

`www: astro examples of scattering`

Scattering and Random Walks

Can we understand photon propagation with isotropic scattering in a simple physical picture?



simple model: **random walk**

between collisions, photons move in straight "steps"

with random displacement $\hat{\ell}$

position after N collisions ("steps") is $\vec{r}_N$

idealizations:

- *step length uniform*: $|\hat{\ell}| = \ell_{\rm mfp}$ mean free path
- *step direction random*: each $\hat{\ell}$ drawn from isotropic distribution and independent of previous steps
- initial condition: start at center, $\vec{r}_0 = 0$

1. first step $\vec{r}_1 = \hat{\ell}$

length $|\vec{r}_1| = \ell_{\rm mfp}$, direction random

average over ensemble of photons:

- $\langle \vec{r}_1 \rangle = \left\langle \vec{\ell} \right\rangle = 0$ *Q: why?*
- $\left\langle r_1^2 \right\rangle = \ell_{\rm mfp}^2$

average positions for *ensemble* of photons is zero

but average distance of *each* photon $\ell_{\rm mfp}$

2. step N has: $\vec{r}_N = \vec{r}_{N-1} + \hat{\ell}$

average over ensemble of photons:

$\langle \vec{r}_N \rangle = \langle \vec{r}_{N-1} \rangle + \left\langle \hat{\ell} \right\rangle = \langle \vec{r}_{N-1} \rangle$

but by recursion

$$\langle \vec{r}_N \rangle = \langle \vec{r}_{N-1} \rangle = \langle \vec{r}_{N-2} \rangle = \ldots = \langle \vec{r}_1 \rangle = 0 \tag{9}$$

$\rightarrow$ ensemble average of photons displacements still 0

as it must be by symmetry

Q: Wut? Does this means photons don't move?

but what about *mean square* displacement?

$$r_N^2 = \vec{r}_N \cdot \vec{r}_N \tag{10}$$

$$= r_{N-1}^2 + 2\hat{\ell} \cdot \vec{r}_{N-1} + \ell_{\rm mfp}^2 \tag{11}$$

average over photon ensemble

$$\left\langle r_N^2 \right\rangle = \left\langle r_{N-1}^2 \right\rangle + 2\left\langle \hat{\ell} \cdot \vec{r}_{N-1} \right\rangle + \ell_{\rm mfp}^2 \tag{12}$$

Q: what is $\left\langle \hat{\ell} \cdot \vec{r}_{N-1} \right\rangle$*?*

$$\left\langle r_N^2 \right\rangle = \left\langle r_{N-1}^2 \right\rangle + 2 \left\langle \hat{\ell} \cdot \vec{r}_{N-1} \right\rangle + \ell_{\rm mfp}^2 \tag{13}$$

each photon scattering direction independent from previous

$\left\langle \hat{\ell} \cdot \vec{r}_{N-1} \right\rangle = \ell_{\rm mfp} r_{N-1} \langle \cos\theta \rangle = 0$

so $\left\langle r_N^2 \right\rangle = \left\langle r_{N-1}^2 \right\rangle + \ell_{\rm mfp}^2$

but this means $\left\langle r_N^2 \right\rangle = N \ell_{\rm mfp}^2$

$\rightarrow$ **each** photon goes r.m.s. distance

$$r_{\rm rms} = \sqrt{\left\langle r_N^2 \right\rangle} = \sqrt{N} \; \ell_{\rm mfp} \tag{14}$$

so imagine photons generated at $r = 0$

and, after scattering, are observed at distance L

Q: number N of scatterings if optically thin? thick?

Photon Random Walks and Optical Depth

if travel distance L by random walk
then after N scatterings $L = \sqrt{N}\ \ell_{\rm mfp}$
but photon optical depth is $\tau = L/\ell_{\rm mfp}$
$\rightarrow$ counts number of mean free paths in length L

optically thick: $\tau \gg 1$
many scattering events $\rightarrow$ *this is a random walk!*
$N \overset{\rm thick}{\approx} \tau^2$

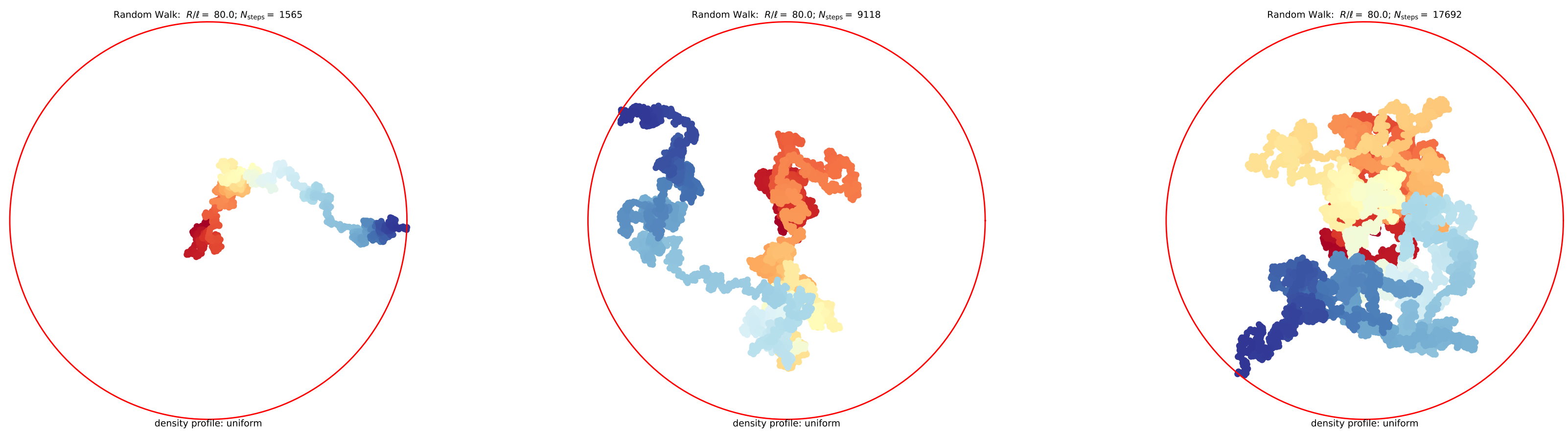
if *optically thin*: $\tau \ll 1$
scattering probability $1 - e^{-\tau} \approx \tau \ll 1$: *not random walk!*
mean number of scatterings over L is $N \overset{\rm thin}{\approx} \tau$

approximate expression good for all τ

$$N \approx \tau + \tau^2 \qquad (15)$$

Photon Escape from Sun: Realizations

easy to simulate! try it!



color shows steps: red at photon birth, blue at escape

not scale! stepsize = $\lambda_{\rm mfp}$ greatly exaggerated here!

Combined Scattering and Absorption

generally, matter can both scatter and absorb photons
transfer equation must include both
for *coherent isotropic scattering* of *thermal radiation*

$$\frac{dI_\nu}{ds} = -\alpha_\nu(I_\nu - B_\nu) - \varsigma_\nu(I_\nu - J_\nu) \tag{16}$$

giving a source function

$$S_\nu = \frac{\alpha_\nu B_\nu + \varsigma_\nu J_\nu}{\alpha_\nu + \varsigma_\nu} \tag{17}$$

a *weighted average* of the two source functions

thus we can write

$$\frac{dI_\nu}{ds} = -(\alpha_\nu + \varsigma_\nu)(I_\nu - S_\nu) \tag{18}$$

with **extinction coefficient** $\alpha_\nu + \varsigma_\nu$

generalize mean free path:

$$\ell_{\mathrm{mfp},\nu} = \frac{1}{\alpha_\nu + \varsigma_\nu} \tag{19}$$

average distance between photon interactions

in random walk picture:
probability of step ending in *absorption*

$$\epsilon_\nu \equiv \alpha_\nu \ell_{\mathrm{mfp},\nu} = \frac{\alpha_\nu}{\alpha_\nu + \varsigma_\nu} \tag{20}$$

and thus step *scattering probability*

$$\varsigma_\nu \ell_{\mathrm{mfp},\nu} = \frac{\varsigma_\nu}{\alpha_\nu + \varsigma_\nu} = 1 - \epsilon_\nu \tag{21}$$

also known as **single scattering albedo**

source function:

$$S_\nu = \epsilon_\nu B_\nu + (1 - \epsilon_\nu) J_\nu \tag{22}$$

Random Walk with Scattering and Absorption

in *infinite medium*: every photon created is eventually absorbed
typical absorption path $\ell_{\text{abs},\nu} = 1/\alpha_\nu$
typical number of scattering events until absorption is

$$N_{\text{scat}} = \frac{\ell_{\text{abs},\nu}}{\ell_{\text{mfp},\nu}} = \frac{\varsigma_\nu + \alpha_\nu}{\alpha_\nu} = \frac{1}{\epsilon_\nu} \tag{23}$$

so typical distance traveled between creation and absorption

$$\ell_* = \sqrt{N_{\text{scat}}}\ell_{\text{mfp},\nu} = \sqrt{\ell_{\text{abs},\nu}\ell_{\text{mfp},\nu}} = \frac{1}{\sqrt{\alpha_\nu(\alpha_\nu + \varsigma_\nu)}} \tag{24}$$

diffusion/thermalization length or *effective mean free path*

What about a *finite medium* of size s?
define optical thicknesses $\tau_{\text{scat}} = \varsigma_\nu s$, $\tau_{\text{abs}} = \alpha_\nu\ s$
and $\tau_* = s/\ell_* = \tau_{\text{scat}}^{1/2}(\tau_{\text{scat}} + \tau_{\text{abs}})^{1/2}$

Q: expected behavior if $\tau_* \ll 1$? $\tau_* \gg 1$?

$\tau_* = s/\ell_*$: path in units of photon travel
until absorption

$\tau_* \ll 1$: *effectively thin* or *translucent*
photons random walk by scattering, seen before absorption
luminosity of thermal source with volume V is

$$L_\nu \stackrel{\text{thin}}{=} 4\pi\alpha_\nu B_\nu V = 4\pi j_\nu(T) V \tag{25}$$

$\tau_* \gg 1$: *effectively tick*
thermally emitted photons scattered then absorbed before seen
expect $I_\nu \to S_\nu \to B_\nu$
rough estimate of luminosity of thermal source:
most emission from "last scattering" surface of area A
where photons travel $s = \ell_*$

$$L_\nu \stackrel{\text{thick}}{\approx} 4\pi\alpha_\nu B_\nu \ell_* A \approx 4\pi\sqrt{\epsilon_\nu}\, B_\nu A \tag{26}$$

Life Inside a Star

In stars:

- nuclear reactions create energy and γ rays deep in the interior (core)
- the energy and radiation escape to the surface after many interactions

How does this occur?

Consider a point at stellar radius r with temperature $T(r)$ having blackbody radiation at T, and matter

Q: what is intensity pattern (i.e., over solid angle) if T is uniform?
Q: what is the pattern more realistically?
Q: what drives the outward energy flow? what impedes it?
Q: relevant length scale(s) for radiation flow?

if $T(r)$ uniform and has *no gradient*,
so are blackbody intensity B and flux T
$\rightarrow$ no net flow of radiation!

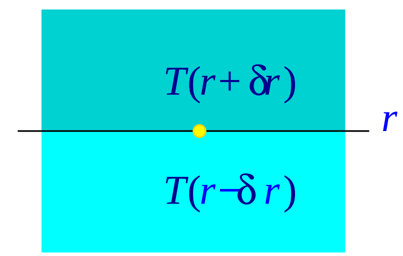
but in real stars: *T decreases with r*

so at r:

- intensity from below greater than from above
- drive net flux outwards
- impeded by scattering and absorption
  on scale $\ell_{\mathrm{mfp},\nu} = (\alpha_\nu + \varsigma_\nu)^{-1}$
- generally $\ell_{\mathrm{mfp},\nu} \ll r$:

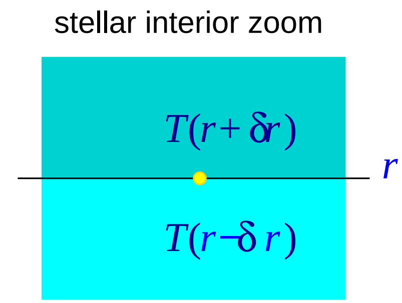
over this scale, see radiation as
mostly isotropic with *small dipole*

stellar interior zoom



Radiative Diffusion: Sketch Rosseland Approximation

small temperature dipole gives *net radiation flux*



$$
\begin{aligned}
F_\nu^{\text{net}} &\sim -\pi \Delta B_\nu \sim -\pi \left[B_\nu(T_{r+\delta r}) - B_\nu(T_{r+\delta r}) \right] \\
&= -\pi \frac{\partial B_\nu}{\partial T} \, \frac{\partial T}{\partial r} \, \delta r \\
&= -\pi \frac{\partial B_\nu}{\partial T} \, \frac{\partial T}{\partial r} \, \ell_{\text{mfp},\nu}
\end{aligned}
$$

So the total flux $F = \int F_\nu^{\text{net}} d\nu$ has

$$F = -\frac{4}{3}\pi \frac{1}{\alpha_{\text{R}}} \frac{\partial_T B}{\partial_r T} \tag{27}$$

- $\vec{F} \propto -\nabla T$: diffusion flux! requires gradient!
- average over ν gives **Rosseland mean absorption coefficient**

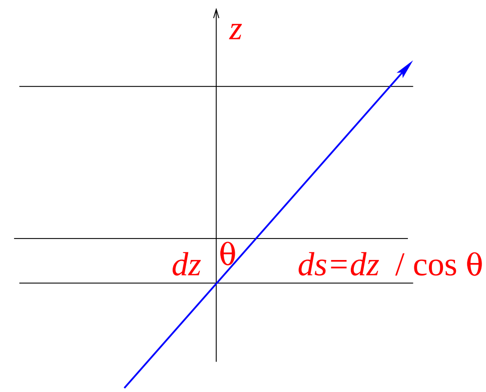
$$\frac{1}{\alpha_{\text{R}}} = \frac{\int (\alpha_\nu + \varsigma_\nu)^{-1} \partial_T B_\nu \; d\nu}{\int \partial_T B_\nu \; d\nu} \tag{28}$$

effective mean free path, weighted by Planck derivative

Director's Cut Extras

Rosseland Approximation in Detail

Imagine a **plane-parallel medium**:
n, ρ, T only depend on z
Think: interior of a star



photon propagation depends only on angle θ
between path direction and $\hat{z}$ *Q: why? why not on* ϕ *too?*

change to variable $\mu = \cos\theta$, and note that
path element $ds = dz/\cos\theta = dz/\mu$, so

$$\mu \frac{\partial I_\nu(z,\mu)}{\partial z} = -(\alpha_\nu + \varsigma_\nu)(I_\nu - S_\nu) \tag{29}$$

note: deep inside a real star like the Sun, $\ell_* \sim 1$ cm $\ll R_\star$
Q: implications?

$\ell_* \sim 1$ cm $\ll R_\star$: rapid thermalization, damping of anisotropy

expect stellar interior to have intensity field that
- changes slowly compared to mean free path
- is nearly isotropic

so to *zeroth order* in ℓ_*, transfer equation

$$I_\nu = S_\nu - \mu \ell_* \frac{\partial I_\nu(z,\mu)}{\partial z} \tag{30}$$

gives

$$I_\nu^{(0)} \approx S_\nu^{(0)}(T) \tag{31}$$

this is angle-independent, so: $J_\nu^{(0)} = S_\nu^{(0)}$ and $I_\nu^{(0)} = S_\nu^{(0)} = B_\nu$

Iterate to get *first-order approximation*

$$I_\nu^{(1)} \approx S_\nu^{(0)} - \mu \ell_* \partial_z I_\nu^{(0)} = B_\nu - \frac{\mu}{\alpha_\nu + \varsigma_\nu} \partial_z B_\nu \tag{32}$$

what angular pattern does this intensity field have? why?

to first order, intensity pattern

$$I_\nu^{(1)} \approx S_\nu^{(0)} - \mu \ell_* \partial_z I_\nu^{(0)} = B_\nu - \frac{\mu}{\alpha_\nu + \varsigma_\nu} \partial_z B_\nu \tag{33}$$

i.e., a dominant isotropic component plus
small correction $\propto \mu = \cos\theta$: a *dipole!*
if T decreases with z, then $\partial_z B_\nu < 0$, and so
intensity brighter downwards (looking into hotter region)

use this find **net specific flux along** z

$$F_\nu(z) = \int I_\nu^{(1)}(z,\mu) \ \cos\theta \ d\Omega = 2\pi \int_{-1}^{+1} I_\nu^{(1)}(z,\mu) \ \mu \ d\mu \tag{34}$$

only the *anisotropic* piece of $I_\nu^{(0)}$ of survives *Q: why?*

$$F_\nu(z) = -\frac{2\pi}{\alpha_\nu + \varsigma_\nu} \partial_z B_\nu \int_{-1}^{+1} \mu^2 \ d\mu \tag{35}$$

$$= -\frac{4\pi}{3(\alpha_\nu + \varsigma_\nu)} \partial_z B_\nu \tag{36}$$

net specific flux along z

$$F_\nu(z) = -\frac{4\pi}{3(\alpha_\nu + \varsigma_\nu)}\partial_z B_\nu = -\frac{4\pi}{3(\alpha_\nu + \varsigma_\nu)}\partial_T B_\nu \; \partial_z T \tag{37}$$

since $B_\nu = B_\nu(T)$

total integrated flux

$$F(z) = \int F_\nu(z)\; d\nu = -\frac{4\pi}{3}\partial_z T \int (\alpha_\nu + \varsigma_\nu)^{-1}\frac{\partial B_\nu}{\partial T}\; d\nu \tag{38}$$

to make pretty, note that

$$\int \partial_T B_\nu \; d\nu = \partial_T \int B_\nu \; d\nu = \partial_T B(T) = \frac{4\pi\sigma T^3}{\pi} \tag{39}$$

and define **Rosseland mean absorption coefficient**

$$\frac{1}{\alpha_{\mathrm{R}}} = \frac{\int (\alpha_\nu + \varsigma_\nu)^{-1}\partial_T B_\nu \; d\nu}{\int \partial_T B_\nu \; d\nu} \tag{40}$$

average effective mean free path, weighted by Planck derivative