Astronomy 501: Radiative Processes

Lecture 11

Sept 16, 2022

Announcements:

- **Problem Set 3 due 5pm today**
- **Problem Set 4 due next Friday**
- PS3 Q2(a) hint: averaging procedure is trivial!

Last time: scattering

Today: changing gears

- heretofore, photon=quantum picture of EM radiation
- now, (re)visit classical picture of EM fields and waves

  Q: regime of applicability?

  Q: classical force on charge q with velocity $\vec{v}$?

  Q: power supplied by EM fields to the charge?

Classical Electromagnetic Radiation

Electromagnetic Forces on Particles

Consider *non-relativistic classical particle*
with mass m, charge q and velocity $\vec{v}$

in an electric field $\vec{E}$ and magnetic field $\vec{B}$
the particle feels Coulomb and Lorentz **forces**

$$\vec{F} = q\ \vec{E} + q\ \frac{\vec{v}}{c} \times \vec{B} \tag{1}$$

units: cgs throughout; has nice property that $[E] = [B]$
ugly SI equations in Extras below

power supplied by EM fields to charge

$$\frac{dU_{\mathrm{mech}}}{dt} = \vec{v} \cdot \vec{F} = q\ \vec{v} \cdot \vec{E} = \frac{d}{dt}\frac{mv^2}{2} \tag{2}$$

no contribution from $\vec{B}$: "magnetic fields do no work"

Q: what if smoothly distributed charge density and velocity field?

Electromagnetic Forces on Continuous Media

consider a medium with charge density ρ_q
and current density $\vec{j} = \rho_q \vec{v}$

by considering an "element" of charge $dq = \rho_q \, dV$
we find **force density**, defined via $d\vec{F} = \vec{f} \, dV$:

$$\vec{f} = \rho_q \, \vec{E} + \frac{\vec{j}}{c} \times \vec{B} \tag{3}$$

and a **power density** supplied by the fields

$$\frac{\partial u_{\text{mech}}}{\partial t} = \vec{j} \cdot \vec{E} \tag{4}$$

note: if medium is a collection of point sources $q_i, \vec{r}_i, \vec{v}_i$

$$\rho_q(\vec{r}) = \sum_i q_i \ \delta(\vec{r} - \vec{r}_i) \tag{5}$$

and current density is

$$\vec{j}(\vec{r}) = \sum_i q_i \ \vec{v}_i \ \delta(\vec{r} - \vec{r}_i) \tag{6}$$

forces control particle responses to fields
now need equations for fields themselves!

Q: sources of electric fields? point source behavior?
Q: sources of magnetic fields? non-sources?
infinite wire behavior?

Maxwell's Equations

Maxwell relates fields to charge and current distributions

in the absence of dielectric media ($\epsilon = 1$)
or permeable media ($\mu = 1$):

$$\begin{aligned}
\nabla \cdot \vec{E} &= 4\pi\rho_q && \text{Coulomb's law} \\
\nabla \cdot \vec{B} &= 0 && \text{no magnetic monopoles} \\
\nabla \times \vec{E} &= -\frac{1}{c}\partial_t \vec{B} && \text{Faraday's law} \\
\nabla \times \vec{B} &= \frac{4\pi}{c}\vec{j} + \frac{1}{c}\partial_t \vec{E} && \text{Ampère's law}
\end{aligned} \tag{7}$$

imagine I know:
- fields $\vec{E}_1, \vec{B}_1$ arising from $\rho_1, \vec{j}_1$
- and fields $\vec{E}_2, \vec{B}_2$ arising from $\rho_2, \vec{j}_2$

now consider case of sources $\rho_1 + \rho_2$ and $\vec{j}_1 + \vec{j}_2$

Q: what are the resulting fields? why?

Maxwell's equations are linear in the fields and sources!
for example: if $\nabla \cdot \vec{E}_1 = \rho_1$ and $\nabla \cdot \vec{E}_2 = \rho_2$
then $\nabla \cdot (\vec{E}_1 + \vec{E}_2) = \rho_1 + \rho_2$

can show: same idea for currents

and thus: **superposition** holds!
sum of sources leads to fields that *sum solutions for each*

Q: divergence of Ampère?

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\partial_t \vec{E} \qquad (8)$$

take divergence of Ampère

$$\partial_t \rho_q + \nabla \cdot \vec{j} = 0 \quad \text{continuity} \tag{9}$$

integrate over volume:

$$\frac{dQ}{dt} = \int \partial_t \rho_q \; dV = -\int \nabla \cdot \vec{j} \; dV \stackrel{\text{Gauss thm}}{=} -\int \vec{j} \cdot d\vec{A} = I_q \tag{10}$$

charge loss from volume is only due to current out

conservation of charge!

now can rewrite power exerted by fields on charges
in terms of fields only *Q: how?*

Field Energy

Power density exerted by fields on charges

$$\frac{\partial u_{\text{mech}}}{\partial t} = \vec{j} \cdot \vec{E} = \frac{1}{4\pi} \left(c\nabla \times \vec{B} - \partial_t \vec{E} \right) \cdot \vec{E} \qquad (11)$$

with clever repeated use of Maxwell,
can recast in this form:

$$\frac{\partial u_{\text{fields}}}{\partial t} + \nabla \cdot \vec{S} = -\frac{\partial u_{\text{mech}}}{\partial t} \qquad (12)$$

where u_{fields} and $\vec{S}$ depend only on the fields
and u_{mech} sums the particle (mechanical) energies

Q: physical significance of eq. (12)?

energy change per unit time

$$\frac{\partial u_{\text{fields}}}{\partial t} + \nabla \cdot \vec{S} = -\frac{\partial u_{\text{mech}}}{\partial t} \tag{13}$$

reminiscent of $\partial_t \rho_q + \nabla \cdot \vec{j} = 0$
$\rightarrow$ an expression of local conservation of energy
where the mechanical energy acts as source/sink

identify **electromagnetic field energy density**

$$u_{\text{fields}} = \frac{E^2 + B^2}{8\pi} \tag{14}$$

i.e., $u_E = E^2/8\pi$, and $u_B = B^2/8\pi$

and **Poynting vector** is *flux of EM energy*

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} \tag{15}$$

this is huge for us ASTR 501 folk! EM flux!
Q: when zero? nonzero? direction?

Maxwell in Vacuo

Now consider a **vacuum = no charges or currents**
Maxwell simplifies to

$$\nabla \cdot \vec{E} = 0 \tag{16}$$

$$\nabla \cdot \vec{B} = 0 \tag{17}$$

$$\nabla \times \vec{E} = -\frac{1}{c}\partial_t \vec{B} \tag{18}$$

$$\nabla \times \vec{B} = \frac{1}{c}\partial_t \vec{E} \tag{19}$$

Q: are there trivial solutions?

Q: are there non-trivial solutions? why?

Q: what scales appear? what doesn't appear? implications?

Electromagnetic Waves

in vacuum ($\rho_q = 0 = \vec{j}$), and in Cartesian coordinates Maxwell's equations imply (PS3):

$$\nabla^2 \vec{E} - \frac{1}{c^2}\partial_t^2 \vec{E} = 0 \tag{20}$$

$$\nabla^2 \vec{B} - \frac{1}{c^2}\partial_t^2 \vec{B} = 0 \tag{21}$$

Q: why is this gorgeous and profound?

Q: natural description?

vacuum Maxwell:

$$\nabla^2 \vec{E} - \frac{1}{c^2}\partial_t^2 \vec{E} = 0 \tag{22}$$

$$\nabla^2 \vec{B} - \frac{1}{c^2}\partial_t^2 \vec{B} = 0 \tag{23}$$

both fields satisfy a **wave equation**
i.e., both fields support (undamped) waves with *speed* c

simplest wave solutions: sinusoids
superposition: arbitrary wave is sum of sinusoids

wave equation invites **Fourier transform** of fields:

$$\vec{E}(\vec{k},\omega) = \frac{1}{(2\pi)^2}\int d^3\vec{r}\ dt\ \ \vec{E}(\vec{r},t)\ e^{-i(\vec{k}\cdot\vec{r}-\omega t)} \tag{24}$$

inverse transformation:

$$\vec{E}(\vec{x},t) = \frac{1}{(2\pi)^2}\int d^3\vec{k}\ d\omega\ \ \vec{E}(\vec{k},\omega)\ e^{i(\vec{k}\cdot\vec{x}-\omega t)} \tag{25}$$

note symmetry between transformation (but sign flip in phase!)

original real-space field can be expressed as

$$\vec{E}(\vec{x},t) = \frac{1}{(2\pi)^2} \int d^3\vec{k}\ d\omega \quad \vec{E}(\vec{k},\omega)\ e^{i(\vec{k}\cdot\vec{x}-\omega t)} \tag{26}$$

expansion in *sum of Fourier modes* with

- **wavevector** $\vec{k}$
  magnitude $k = 2\pi/\lambda$, propagation direction $\hat{n} = \vec{k}/k$
- **angular frequency** $\omega = 2\pi\ \nu$

apply wave equation to Fourier expansion:

$$\begin{aligned} \nabla^2\vec{E} - \frac{1}{c^2}\partial_t^2\vec{E} &= -\frac{1}{(2\pi)^2 c^2} \int d^3\vec{k}\ d\omega\ (c^2k^2 - \omega^2)\ \vec{E}(\vec{k},\omega)\ e^{i(\vec{k}\cdot\vec{x}-\omega t)} \\ &= 0 \end{aligned}$$

for non-trivial solutions with $\vec{E} \neq 0$,
this requires $\omega^2 = c^2k^2$, or **vacuum dispersion relation**

$$\omega = ck \tag{27}$$

i.e., wave solutions require constant phase velocity $v_\phi = \omega/k = c$

Maxwell and Fourier Modes

We have seen: wave equation demands $\omega = ck$
But Maxwell equations impose further constraints

Consider arbitrary Fourier modes
$\vec{E} = E_0 \ e^{i(\vec{k}\cdot\vec{x}-\omega t)} \ \hat{a}_1$, and $\vec{B} = B_0 \ e^{i(\vec{k}\cdot\vec{x}-\omega t)} \ \hat{a}_2$

Maxwell equations in vacuum impose conditions:
for example, Coulomb's law $\nabla \cdot \vec{E} = 0$ implies

$$\vec{k} \cdot \vec{E} = 0 \tag{28}$$

or equivalently $\hat{n} \cdot \hat{a}_1 = 0$

similarly, no monopoles requires

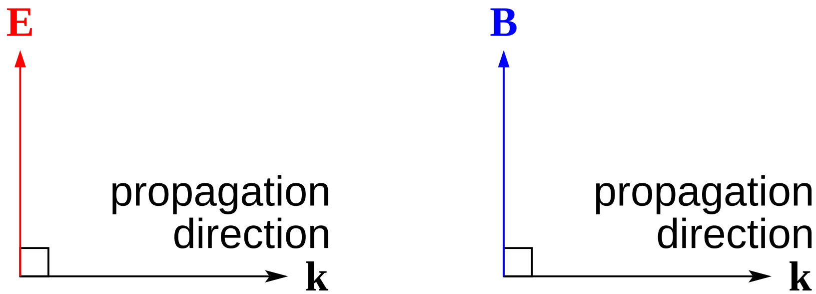
$$\vec{k} \cdot \vec{B} = 0 \qquad \hat{n} \cdot \hat{a}_2 = 0 \tag{29}$$

Q: what does this mean physically for the waves?

we found $\vec{k} \cdot \vec{E} = \vec{k} \cdot \vec{B} = 0$

$\rightarrow$ propagation orthogonal to field vectors

$\Rightarrow$ *EM waves are transverse*



Faraday's law requires $\omega\vec{B} = c\vec{k} \times \vec{E}$, or

$$\vec{B} = \frac{c\vec{k}}{\omega} \times \vec{E} = \hat{n} \times \vec{E} \tag{30}$$

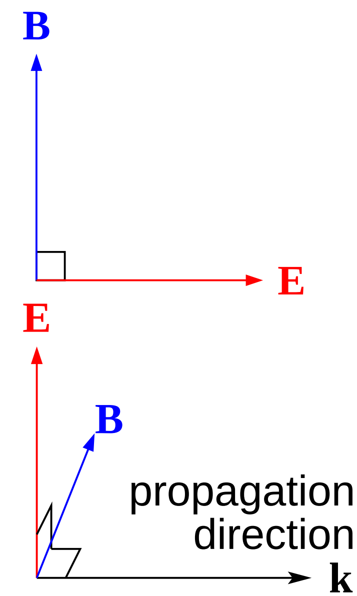
and Ampère's law gives $\vec{E} = -\hat{n} \times \vec{B}$

Q: what do these conditions imply for the waves?

Faraday's law gives $\boxed{\vec{B} = \hat{n} \times \vec{E}}$, so

$$\vec{E} \cdot \vec{B} = \vec{E} \cdot (\hat{n} \times \vec{E}) = 0$$

$\Rightarrow$ $\vec{E}$ *and* $\vec{B}$ *are orthogonal to each other!*



Faraday also implies

$$|B|^2 = \hat{n}^2 |E|^2 - |\hat{n} \cdot \vec{E}|^2 = |E|^2 \tag{31}$$

using vector identity $(\hat{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \hat{a} \cdot \vec{c}\, \vec{b} \cdot \vec{d} - \hat{a} \cdot \vec{d}\, \vec{b} \cdot \vec{c}$

we have: $E_0 = B_0$: *field amplitudes are equal*

which in turn means: $\hat{a}_2 = \hat{n} \times \hat{a}_1$, and $\hat{a}_1 \cdot \hat{a}_2 = 0$
$\rightarrow$ $(\hat{n}, \hat{a}_1, \hat{a}_2)$ form an *orthogonal basis*

Monochromatic Plane Wave: Time Averaging

at a given point in space, field amplitudes vary
sinusoidally with time $\rightarrow$ energy density and flux also sinusoidal
but we are interested in timescales $\gg \omega^{-1}$:
$\rightarrow$ take *time averages*

Useful to use *complex* field amplitudes
then take *real part* to get physical component

handy theorem: for $A(t) = \mathcal{A}e^{i\omega t}$ and $B(t) = \mathcal{B}e^{i\omega t}$
i.e., same time dependence, then time-averaged products

$$\langle \text{Re}A(t)\ \text{Re}B(t)\rangle = \frac{1}{2}\text{Re}(\mathcal{A}\mathcal{B}^*) = \frac{1}{2}\text{Re}(\mathcal{A}^*\mathcal{B}) \tag{32}$$

Monochromatic Plane Wave: Energy, Flux

time-averaged Poynting flux amplitude

$$\langle S \rangle = \frac{c}{8\pi} \mathrm{Re}(E_0 B_0^*) = \frac{c}{8\pi} |E_0|^2 = \frac{c}{8\pi} |B_0|^2 \qquad (33)$$

relates intensity and field strength

time-averaged energy density

$$\langle u \rangle = \frac{|E_0|^2}{8\pi} = \frac{|B_0|^2}{8\pi} \qquad (34)$$

and so $\left\langle \vec{S} \right\rangle = c \left\langle u \right\rangle \hat{n}$

Q: given wave direction $\vec{n}$, degrees of freedom in $\vec{E}, \vec{B}$?

Director's Cut Extras

Electromagentism in SI Units

Sadly, unit conversion in between SI and cgs is a stain on the otherwise beautiful subject of E&M

Here we summarize how the fundamental equations appear in SI units

Coulomb and Lorentz forces in SI

$$\vec{F} = q\vec{E} + q\,\vec{v} \times \vec{B} \quad (35)$$

$$\vec{f} = \rho_q\,\vec{E} + \vec{j} \times \vec{B} \quad (36)$$

note this means that E and B have different units!

Maxwell's equations in SI:

$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{\rho_q}{\epsilon_0} && \text{Coulomb's law} \\ \nabla \cdot \vec{B} &= 0 && \text{no magnetic monopoles} \\ \nabla \times \vec{E} &= -\partial_t \vec{B} && \text{Faraday's law} \\ \nabla \times \vec{B} &= \mu_0 \vec{j} + \mu_0 \epsilon_0 \, \partial_t \vec{E} && \text{Ampère's law} \end{aligned} \tag{37}$$

and we find that $\epsilon_0 \mu_0 = 1/c^2$

field energy density (note the ghastly lack of symmetry!)

$$u_{\text{fields}} = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \tag{38}$$

i.e., $u_E = \epsilon_0 E^2/2$, and $u_B = B^2/2\mu_0$

and **Poynting vector** is *flux of EM energy*

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \tag{39}$$