

# Astronomy 501: Radiative Processes

Lecture 12

Sept 19, 2022

Announcements:

- **Problem Set 4 due Friday**

Last time: began classical EM radiation

*Q: energy density?*

*Q: Poynting vector?*

**Poynting vector** is *flux of EM energy*

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} \quad (1)$$

**electromagnetic field energy density**

$$u_{\text{fields}} = \frac{E^2 + B^2}{8\pi} \quad (2)$$

i.e.,  $u_E = E^2/8\pi$ , and  $u_B = B^2/8\pi$

Note: can guess energy density form  
via “what else can it be” method

- fields  $\vec{E}$  and  $\vec{B}$  are vectors
- but energy density  $u$  is a positive scalar
- have to take dot products
- only positive definite options are  $\vec{E} \cdot \vec{E} = E^2$  and  $B^2$

vacuum Maxwell gives:

$$\nabla^2 \vec{E} - \frac{1}{c^2} \partial_t^2 \vec{E} = 0 \quad (3)$$

$$\nabla^2 \vec{B} - \frac{1}{c^2} \partial_t^2 \vec{B} = 0 \quad (4)$$

both fields satisfy a **wave equation**

i.e., both fields support (undamped) waves with **speed  $c$**

simplest wave solutions: sinusoids

superposition: arbitrary wave is sum of sinusoids

wave equation invites **Fourier transform** of fields:

$$\vec{E}(\vec{k}, \omega) = \frac{1}{(2\pi)^2} \int d^3\vec{r} dt \vec{E}(\vec{x}, t) e^{-i(\vec{k}\cdot\vec{r}-\omega t)} \quad (5)$$

inverse transformation:

$$\omega \quad \vec{E}(\vec{x}, t) = \frac{1}{(2\pi)^2} \int d^3\vec{k} d\omega \vec{E}(\vec{k}, \omega) e^{i(\vec{k}\cdot\vec{r}-\omega t)} \quad (6)$$

note symmetry between transformation (but sign flip in phase!)

original real-space field can be expressed as

$$\vec{E}(\vec{x}, t) = \frac{1}{(2\pi)^2} \int d^3\vec{k} d\omega \vec{E}(\vec{k}, \omega) e^{i(\vec{k}\cdot\vec{r}-\omega t)} \quad (7)$$

expansion in *sum of Fourier modes* with

- **wavevector**  $\vec{k}$   
magnitude  $k = 2\pi/\lambda$ , propagation direction  $\hat{n} = \vec{k}/k$
- **angular frequency**  $\omega = 2\pi \nu$

apply wave equation to Fourier expansion:

$$\begin{aligned} \nabla^2 \vec{E} - \frac{1}{c^2} \partial_t^2 \vec{E} &= -\frac{1}{(2\pi)^2 c^2} \int d^3\vec{k} d\omega (c^2 k^2 - \omega^2) \vec{E}(\vec{k}, \omega) e^{i(\vec{k}\cdot\vec{r}-\omega t)} \\ &= 0 \end{aligned}$$

for non-trivial solutions with  $\vec{E} \neq 0$ ,

this requires  $\omega^2 = c^2 k^2$ , or **vacuum dispersion relation**

$$\Rightarrow \omega = ck \quad (8)$$

i.e., wave solutions require constant phase velocity  $v_\phi = \omega/k = c$

## Maxwell and Fourier Modes

We have seen: wave equation demands  $\omega = ck$   
But Maxwell equations impose further constraints

Consider arbitrary Fourier modes

$$\vec{E} = E_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \hat{a}_1, \text{ and } \vec{B} = B_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \hat{a}_2$$

Maxwell equations in vacuum impose conditions:  
for example, Coulomb's law  $\nabla \cdot \vec{E} = 0$  implies

$$\vec{k} \cdot \vec{E} = 0 \tag{9}$$

or equivalently  $\hat{n} \cdot \hat{a}_1 = 0$

similarly, no monopoles requires

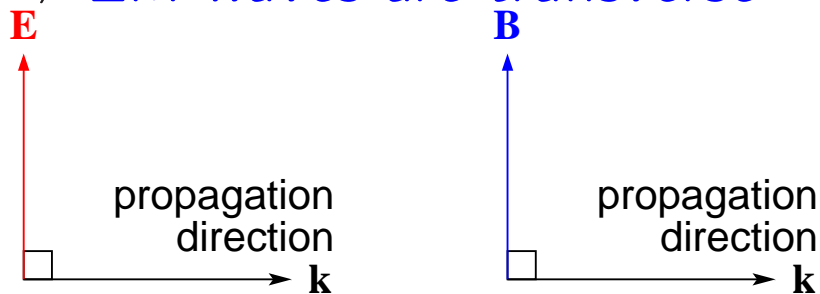
$$\vec{k} \cdot \vec{B} = 0 \quad \hat{n} \cdot \hat{a}_2 = 0 \tag{10}$$

*Q: what does this mean physically for the waves?*

we found  $\vec{k} \cdot \vec{E} = \vec{k} \cdot \vec{B} = 0$

→ propagation orthogonal to field vectors

⇒ *EM waves are transverse*



Faraday's law requires  $\omega \vec{B} = c \vec{k} \times \vec{E}$ , or

$$\vec{B} = \frac{c \vec{k}}{\omega} \times \vec{E} = \hat{n} \times \vec{E} \quad (11)$$

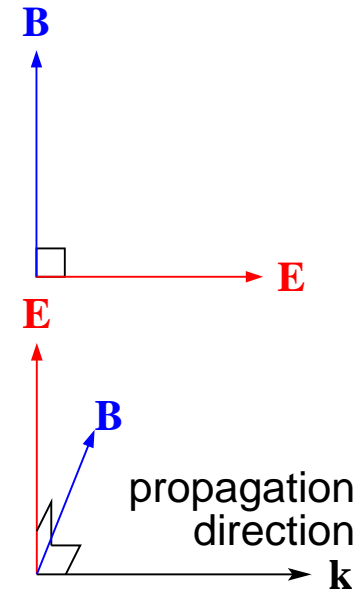
and Ampère's law gives  $\vec{E} = -\hat{n} \times \vec{B}$

○ Q: *what do these conditions imply for the waves?*

Faraday's law gives  $\vec{B} = \hat{n} \times \vec{E}$ , so

$$\vec{E} \cdot \vec{B} = \vec{E} \cdot (\hat{n} \times \vec{E}) = 0$$

$\Rightarrow \vec{E}$  and  $\vec{B}$  are orthogonal to each other!



Faraday also implies

$$|B|^2 = \hat{n}^2 |E|^2 - |\hat{n} \cdot \vec{E}|^2 = |E|^2 \quad (12)$$

using vector identity  $(\hat{a} \times \hat{b}) \cdot (\hat{c} \times \hat{d}) = \hat{a} \cdot \hat{c} \hat{b} \cdot \hat{d} - \hat{a} \cdot \hat{d} \hat{b} \cdot \hat{c}$

we have:  $E_0 = B_0$ : *field amplitudes are equal*

which in turn means:  $\hat{a}_2 = \hat{n} \times \hat{a}_1$ , and  $\hat{a}_1 \cdot \hat{a}_2 = 0$   
 $\rightarrow (\hat{n}, \hat{a}_1, \hat{a}_2)$  form an *orthogonal basis*

## Monochromatic Plane Wave: Time Averaging

at a given point in space, field amplitudes vary sinusoidally with time  $\rightarrow$  energy density and flux also sinusoidal but we are interested in timescales  $\gg \omega^{-1}$ :  
 $\rightarrow$  take *time averages*

Useful to use *complex* field amplitudes  
then take *real part* to get physical component

handy theorem: for  $A(t) = \mathcal{A}e^{i\omega t}$  and  $B(t) = \mathcal{B}e^{i\omega t}$   
i.e., same time dependence, then time-averaged products

$$\langle \text{Re}A(t) \text{Re}B(t) \rangle = \frac{1}{2}\text{Re}(\mathcal{A}\mathcal{B}^*) = \frac{1}{2}\text{Re}(\mathcal{A}^*\mathcal{B}) \quad (13)$$



## Monochromatic Plane Wave: Energy, Flux

time-averaged Poynting flux amplitude

$$\langle S \rangle = \frac{c}{8\pi} \operatorname{Re}(E_0 B_0^*) = \frac{c}{8\pi} |E_0|^2 = \frac{c}{8\pi} |B_0|^2 \quad (14)$$

relates intensity and field strength

time-averaged energy density

$$\langle u \rangle = \frac{|E_0|^2}{8\pi} = \frac{|B_0|^2}{8\pi} \quad (15)$$

and so  $\langle \vec{S} \rangle = c \langle u \rangle \hat{n}$

◦ Q: given wave direction  $\vec{n}$ , degrees of freedom in  $\vec{E}, \vec{B}$ ?

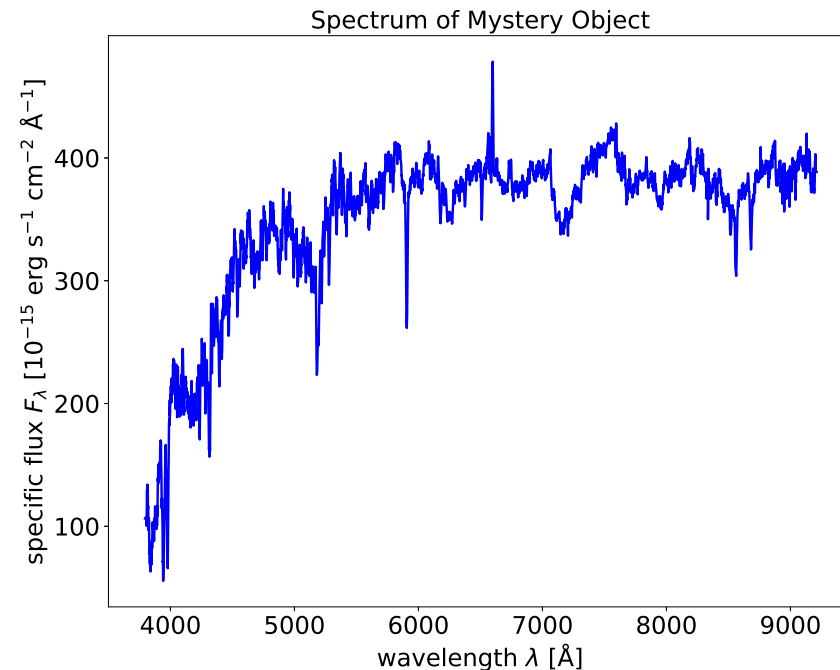
# Intermission: Problem Set 3 Mysteries Solved

## Mystery Object spectrum

Q: main features?

Q: implications for source:  
color? light source?

Q: what is it?



**the Sun at 211Å:** Fe iv = Fe<sup>+12</sup>

Q: EM region? implications of Fe iv?

www: SDO image, movie, compare at other wavelengths

Q: main features? optically thick or thin?

Q: implications for source?

# Polarization

EM waves propagating *in a particular direction*  $\hat{n}$

must be transverse  $\vec{k} \cdot \vec{E} = \hat{n} \cdot \vec{E} = 0$

→ nonzero  $\vec{E}$  components lie in plane  $\perp$  to  $\hat{n}$

*two independent components*

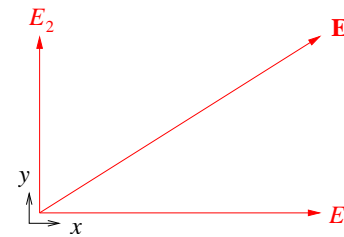
for convenience: choose coordinates where  $\hat{n} = \hat{z}$

so fields are in transverse plane  $x - y$   
physical electric vector is *real part* of

$$\vec{E} = (E_1 \hat{x} + E_2 \hat{y}) e^{-i\omega t} \quad (16)$$

complex amplitudes can be written

$$E_1 = \mathcal{E}_1 e^{i\phi_1} \quad E_2 = \mathcal{E}_2 e^{i\phi_2} \quad (17)$$



Q: *but wait—what about the magnetic field?*

transverse electric field has two independent components  
but once  $\vec{E}$  determined, then  $\vec{B} = \hat{n} \times \vec{E}$   
at every point along sightline  $\hat{n}$ , magnetic  $\perp$  electric  
 $\Rightarrow$  *no additional degrees of freedom for  $\vec{B}$*

monochromatic plane wave *has two independent components*

consider plane at fixed  $z = \hat{n} \cdot \vec{r}$ , say  $z = 0$   
imagine a detector in the plane, with  $x - y$  axes

the two *physical* components of the  $\vec{E}$  field  
are *independent*, and evolve along the detector axes as

$$E_x = \mathcal{E}_1 \cos(\omega t - \phi_1) \quad E_y = \mathcal{E}_2 \cos(\omega t - \phi_2) \quad (18)$$

In our  $x - y$  detector plane,  
the independent  $\vec{E}$  field component sare:

$$E_x = \mathcal{E}_1 \cos(\omega t - \phi_1) \quad E_y = \mathcal{E}_2 \cos(\omega t - \phi_2) \quad (19)$$

with  $E_1, E_2$  can take any values, and  $\phi_1, \phi_2$  independent  
but only difference  $\phi_1 - \phi_2$  can be important  
→ a total of *3 independent parameters* describe the fields

Q:  $\vec{E}$  time evolution if  $E_1$  and  $E_2$  can differ, but  $\phi_1 - \phi_2 = 0$ ?

Q: same but  $\phi_1 - \phi_2 = \pi$ ?

# Linear Polarization

For  $\phi_1 - \phi_2 = 0$ , we have

$$E_x = \mathcal{E}_1 \cos(\omega t - \phi_1) \quad E_y = \mathcal{E}_2 \cos(\omega t - \phi_1) = \frac{\mathcal{E}_2}{\mathcal{E}_1} E_x \quad (20)$$

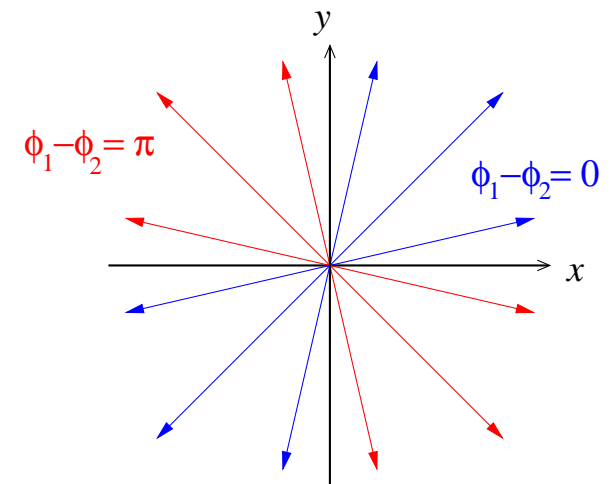
fields share same sign and same sinusoidal time dependence

$\vec{E}$  sweeps out *line with positive slope* in  $x - y$  plane

→ **linear polarization**

For  $\phi_1 - \phi_2 = \pi$ , fields share time dependence but have **opposite sign**

→ linear polarization with negative slope



Q: what is  $\vec{E}$  time dependence if  $\mathcal{E}_1 = \mathcal{E}_2$  but  $\phi_1 - \phi_2 = \pi/2? -\pi/2$

# Circular Polarization

if  $\mathcal{E}_1 = \mathcal{E}_2$  but  $\phi_1 - \phi_2 = \pi/2$

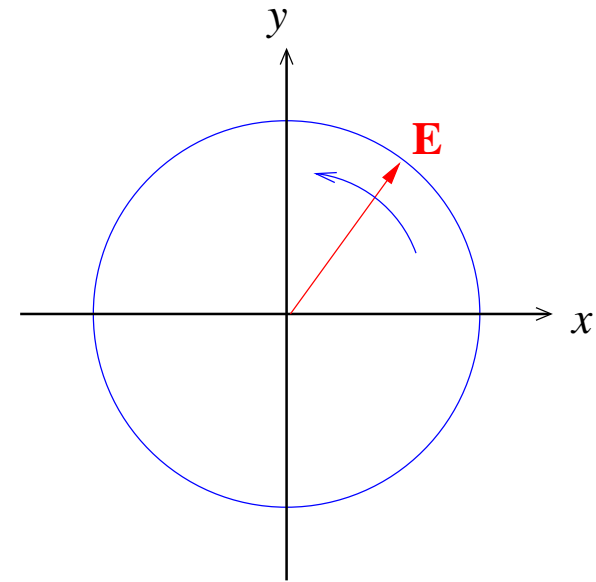
$$E_x = \mathcal{E}_1 \cos(\omega t - \phi_1) \quad E_y = \mathcal{E}_1 \sin(\omega t - \phi_1)$$

$\vec{E}$  sweeps counterclockwise circle  
as seen approaching observer

⇒ **circular polarization**

Engineering: “*lefthanded*” circular polarization

→ but using righthand rule: *positive helicity*



if  $\mathcal{E}_1 = \mathcal{E}_2$  but  $\phi_1 - \phi_2 = -\pi/2$

→ “*righthand*” circular polarization, or *negative helicity*

in the most general case:  $\mathcal{E}_1 \neq \mathcal{E}_2$  and  $\phi_1 - \phi_2$  arbitrary

Q: what is  $\vec{E}$  time dependence?

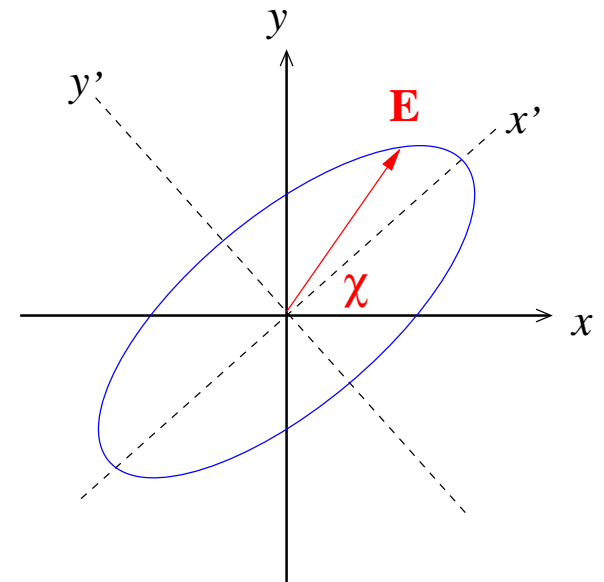
# Elliptical Polarization

in the general case

$$E_x = \mathcal{E}_1 \cos(\omega t - \phi_1) \quad E_y = \mathcal{E}_2 \cos(\omega t - \phi_2)$$

intuitively, blends linear and circular features:

→ **elliptical polarization**



ellipse *orientation* fixed by  $\mathcal{E}_1 - \mathcal{E}_2$  difference

ellipse *eccentricity* and *helicity* fixed by  $\phi_1 - \phi_2$  difference

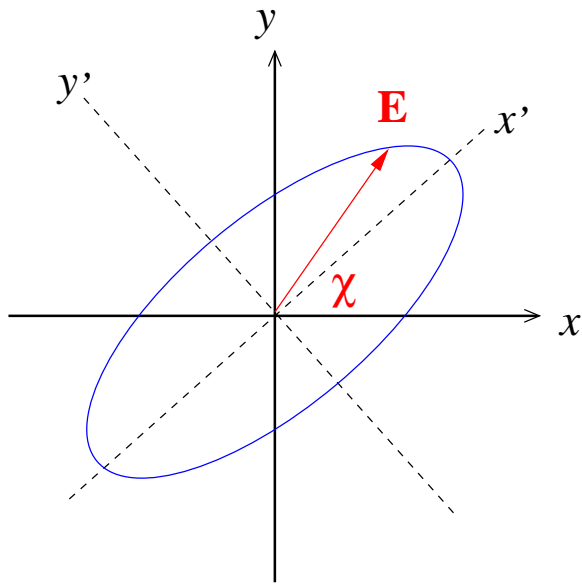
in coordinates  $(x', y')$  rotated to align with *principal axes*

$$E'_x = \mathcal{E}_0 \cos \beta \cos(\omega t) \quad E'_y = -\mathcal{E}_0 \sin \beta \sin(\omega t)$$

for some  $\beta \in [-\pi/2, +\pi/2]$

Q: *evolution* if  $\beta > 0$ ?





$$E'_x = \mathcal{E}_0 \cos \beta \cos(\omega t) \quad E'_y = -\mathcal{E}_0 \sin \beta \sin(\omega t)$$

principle axes:  $\mathcal{E}_0 \cos \beta$  and  $\mathcal{E}_0 \sin \beta$

if  $\beta \in [0, \pi/2]$ : ellipse sweeps clockwise

→ “*righthanded*” elliptical polarization, *negative helicity*

if  $\beta \in [-\pi/2, 0]$ : “*lefthanded*”, *positive helicity*

Q: what  $\beta(s)$  give complete linear polarization? circular?

we want to relate  $x - y$  **field parameters**

$\mathcal{E}_1, \mathcal{E}_2, \phi_1, \phi_2$

to  $x' - y'$  **principle axes parameters**  $\mathcal{E}_0, \beta, \chi$

rotate  $x - y$  components by angle  $\chi$

$$E_x = \mathcal{E}_0 (\cos \beta \cos \chi \cos \omega t + \sin \beta \sin \chi \sin \omega t)$$

$$E_y = \mathcal{E}_0 (\cos \beta \sin \chi \cos \omega t - \sin \beta \cos \chi \sin \omega t)$$

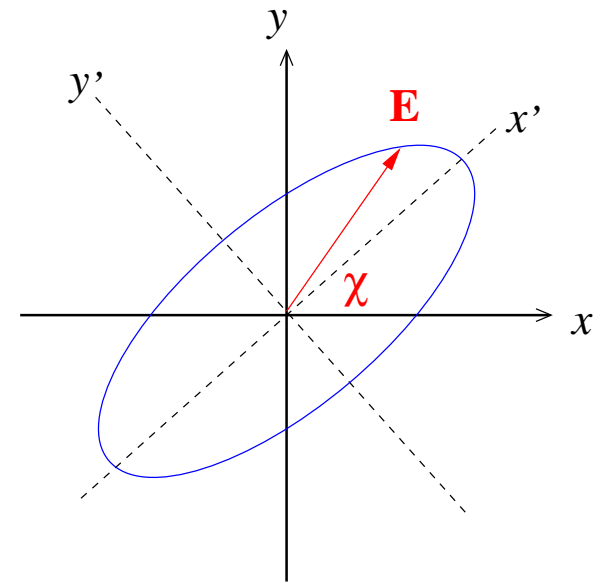
matching to, e.g.,  $E_x = \mathcal{E}_1 \cos(\omega t - \phi_1)$ :

$$\mathcal{E}_1 \cos \phi_1 = \mathcal{E}_0 \cos \beta \cos \chi \quad (21)$$

$$\mathcal{E}_1 \sin \phi_1 = \mathcal{E}_0 \sin \beta \sin \chi \quad (22)$$

$$\mathcal{E}_2 \cos \phi_2 = \mathcal{E}_0 \cos \beta \sin \chi \quad (23)$$

$$\mathcal{E}_2 \sin \phi_2 = -\mathcal{E}_0 \sin \beta \cos \chi \quad (24)$$



Q: how can we determine polarization by intensity measurements with a polarimeters?

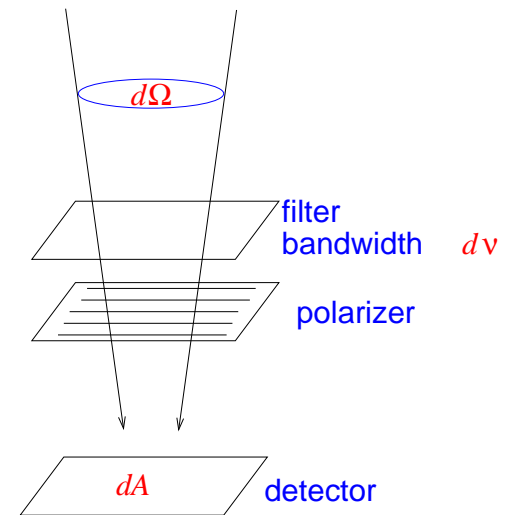
Introduce *polarizer*

can *rotate* polarizer:

→ measure  $I_x$ ,  $I_y$ , and  $45^\circ$  rotated  $I_{x'}$ ,  $I_{y'}$

can use circular polarizers to measure

→ positive and negative circular polarization  $I_+$ ,  $I_-$



combine: **Stokes parameters**

$$I = I_x + I_y \quad (25)$$

$$Q = I_x - I_y \quad (26)$$

$$U = I_{x'} - I_{y'} \quad (27)$$

$$V = I_+ - I_- \quad (28)$$

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Q: what physically is each? can more than one of  $Q, U, V$  be nonzero? what does that correspond to?

Q: range of values for  $Q$ ?  $U$ ?  $V$ ? are they all independent?

## Stokes Parameters

for *monochromatic waves*, Stokes parameters related to  $\mathcal{E}_1, \mathcal{E}_2, \phi_1, \phi_2$  and  $\mathcal{E}_0, \beta, \chi$  bases:

$$I = \mathcal{E}_1^2 + \mathcal{E}_2^2 = \mathcal{E}_0^2 \quad (29)$$

$$Q = \mathcal{E}_1^2 - \mathcal{E}_2^2 = \mathcal{E}_0^2 \cos 2\beta \cos 2\chi \quad (30)$$

$$U = 2\mathcal{E}_1\mathcal{E}_2 \cos(\phi_1 - \phi_2) = \mathcal{E}_0^2 \cos 2\beta \sin 2\chi \quad (31)$$

$$V = 2\mathcal{E}_1\mathcal{E}_2 \sin(\phi_1 - \phi_2) = \mathcal{E}_0^2 \sin 2\beta \quad (32)$$

and thus

$$\mathcal{E}_0 = \sqrt{I} \quad (33)$$

$$\sin 2\beta = V/I \quad (34)$$

$$\tan 2\chi = U/Q \quad (35)$$

since wave has 3 independent parameters,

Stokes parameters must be *related*

$$I^2 = Q^2 + U^2 + V^2 \quad (36)$$

## Quasi-Monochromatic Waves

natural light generally **not a pure monochromatic wave** with a single, definite, complete state of polarization

rather: a *superposition* of components with many polarizations

consider wave with *slowly varying* amplitudes and phases

$$E_1(t) = \mathcal{E}_1(t) e^{i\phi_1(t)} ; \quad E_2(t) = \mathcal{E}_2(t) e^{i\phi_2(t)} \quad (37)$$

“slow”: wave looks completely polarized on timescale  $\omega^{-1}$   
but amplitudes and phases drift over intervals  $\Delta t \gg \omega^{-1}$   
→ polarization changes

but also wave is *no longer monochromatic*

frequency spread: “*bandwidth*”  $\Delta\omega \sim 1/\Delta t \ll \omega$

→ *quasi-monochromatic wave*

Q: *effect on Stokes?*

# Stokes Parameters for Quasi-Monochromatic Light

real measurements represent **averages** over timescales during which polarization can change

Stokes parameters become averages

$$I = \langle E_1 E_1^* \rangle + \langle E_2 E_2^* \rangle = \langle \mathcal{E}_1^2 + \mathcal{E}_2^2 \rangle \quad (38)$$

$$Q = \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle = \langle \mathcal{E}_1^2 - \mathcal{E}_2^2 \rangle \quad (39)$$

$$U = \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle = 2 \langle \mathcal{E}_1 \mathcal{E}_2 \cos(\phi_1 - \phi_2) \rangle \quad (40)$$

$$V = -i (\langle E_1 E_2^* \rangle - \langle E_2 E_1^* \rangle) = 2 \langle \mathcal{E}_1 \mathcal{E}_2 \sin(\phi_1 - \phi_2) \rangle \quad (41)$$

but for quasi-monochromatic waves

$$I^2 \geq Q^2 + U^2 + V^2 \quad (42)$$

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- quasi-monochromatic polarization is still in general *elliptical*
  - but drifts can reduce degree of polarization

$$I^2 \geq Q^2 + U^2 + V^2 \quad (43)$$

- maximum polarization when equality holds: *completely elliptically polarized*
- minimum when  $Q = U = V = 0$ : *unpolarized*
- arbitrary wave is *partially polarized*

useful to define *polarized* intensity

$$I_{\text{pol}} = Q^2 + U^2 + V^2 \quad (44)$$

and since  $I_{\text{pol}} \leq I$ , define fractional **degree of polarization**

$$\Pi \equiv \frac{I_{\text{pol}}}{I} = \frac{\sqrt{Q^2 + U^2 + V^2}}{I} \quad (45)$$

note: can always decompose Stokes parameters

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} I - I_{\text{pol}} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} I_{\text{pol}} \\ Q \\ U \\ V \end{pmatrix} \quad (46)$$

sum of unpolarized and polarized components



## Superposition and Stokes

consider composite wave that is superposition of many independent waves

electric field components are given by **superposition**

$$E_1 = \sum_k E_1^{(k)} \quad ; \quad E_2 = \sum_k E_2^{(k)} \quad (47)$$

each term  $k$  of which has different phase

PS4: phases specified, can calculate sum explicitly

but generally, *phases are random*

so field products average out phases from different waves

$$\langle E_i E_j^* \rangle = \sum_k \sum_\ell \langle E_i^{(k)} E_j^{(\ell)*} \rangle = \sum_k \langle E_i^{(k)} E_i^{(k)*} \rangle \quad (48)$$

but due to this averaging, *Stokes parameters are additive*

$$I = \sum_k I^{(k)} \quad (49)$$

$$Q = \sum_k Q^{(k)} \quad (50)$$

$$U = \sum_k U^{(k)} \quad (51)$$

$$V = \sum_k V^{(k)} \quad (52)$$