Astronomy 501: Radiative Processes

Lecture 13

Sept 21, 2022

Announcements:

- **Problem Set 4 due Friday**
- Office Hours: after class or by appointment
- Physics Colloquium today 4pm: Tracy Slatyer, MIT
  Dark Matter!

Last time: classical EM waves

Q: connection among wave direction $\hat{n}$, $\vec{E}$, *and* $\vec{B}$*?*

polarization – depends on amplitude and phase of $\vec{E}$ components
  in plane transverse to wave direction

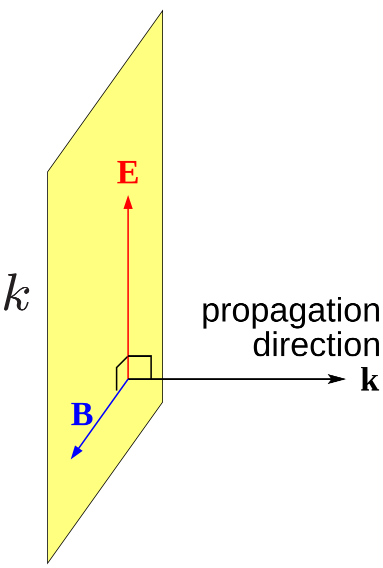
Q: under what conditions do we get linear polarization?

Q: under what conditions do we get circular polarization?

Q: what is polarization for general case?

EM waves in vacuum:

- dispersion: $\omega = ck$, $k = 2\pi/\lambda$
- transverse: $\vec{E} \cdot \vec{k} = 0$ and $\vec{B} \cdot \vec{k} = 0$
- $\|\vec{E}\| = \|\vec{B}\|$, and E, B phases are same
- with propagation direction unit vector $\vec{n} = \vec{k}/k$

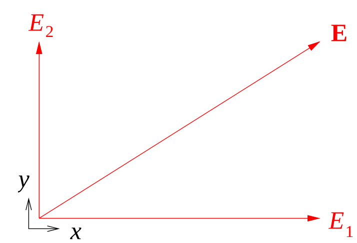


$$\vec{B} = \hat{n} \times \vec{E} = 0$$

so $(\hat{k}, \vec{E}, \vec{B})$ form orthogonal basis

in transverse plane $x - y$
physical electric vector is *real part* of

$$\vec{E} = (E_1\ \hat{x} + E_2\ \hat{y})\ e^{-i\omega t} \qquad (1)$$

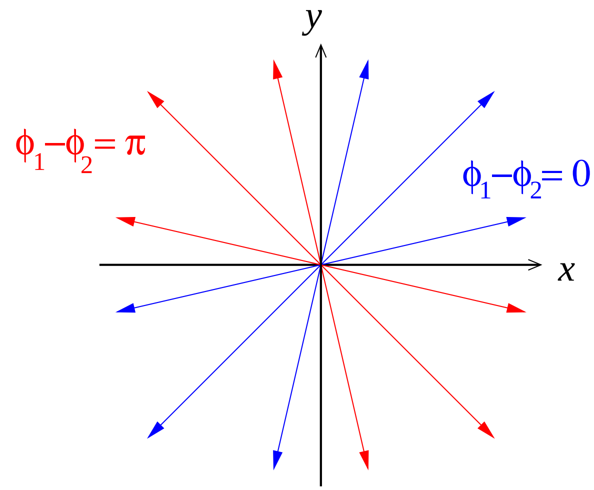


complex amplitudes can be written

$$E_1 = \mathcal{E}_1\ e^{i\phi_1} \quad E_2 = \mathcal{E}_2\ e^{i\phi_2} \qquad (2)$$

For $\phi_1 - \phi_2 = 0$ or π:
linear polarization

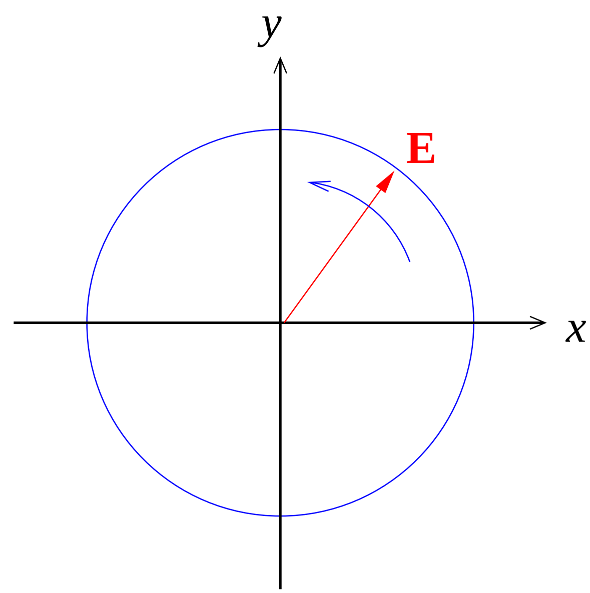
slope set by $\mathcal{E}_1/\mathcal{E}_2$ ratio



if $\mathcal{E}_1 = \mathcal{E}_2$ and $\phi_1 - \phi_2 = \pm\pi/2$

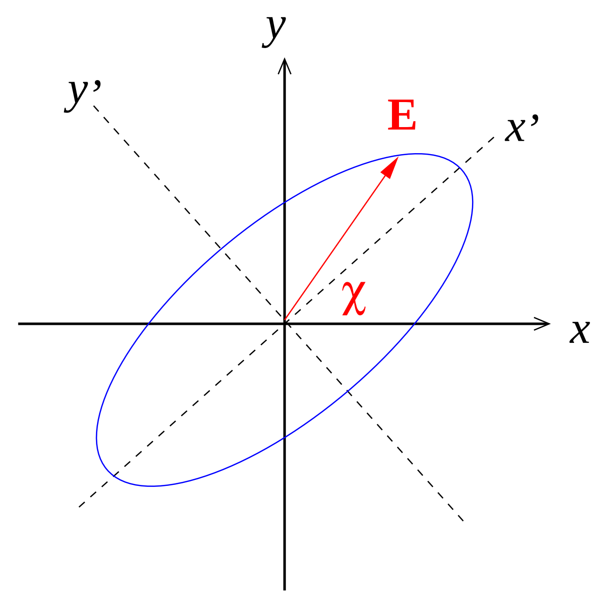
$$E_x = \mathcal{E}_1 \cos(\omega t - \phi_1) \quad E_y = \pm\mathcal{E}_1 \sin(\omega t - \phi_1)$$

$\vec{E}$ sweeps out circle
as seen approaching observer
$\Rightarrow$ **circular polarization**
righthand/lefthand, or positive/negative helicity



Q: what happens in general case of $\mathcal{E}_1 \neq \mathcal{E}_2$ and $\phi_1 \neq \phi_2$?

Elliptical Polarization



in the general case

$$E_x = \mathcal{E}_1 \ \cos(\omega t - \phi_1) \quad E_y = \mathcal{E}_2 \ \cos(\omega t - \phi_2)$$

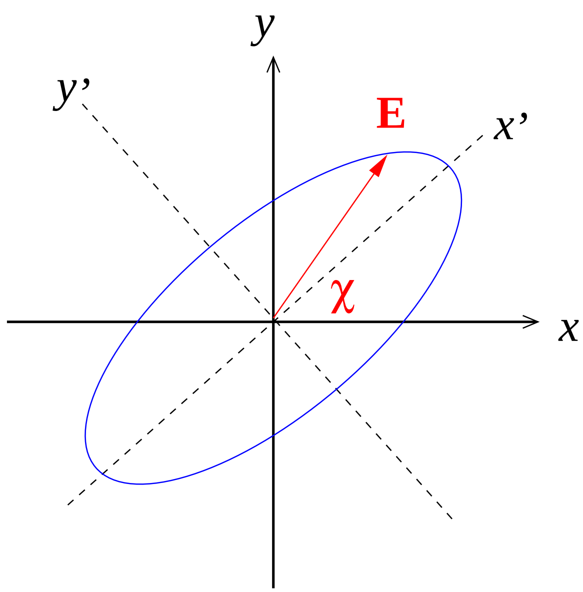
intuitively, blends linear and circular features:
$\rightarrow$ **elliptical polarization**

ellipse *orientation* fixed by $\mathcal{E}_1 - \mathcal{E}_2$ difference
ellipse *eccentricity* and *helicity* fixed by $\phi_1 - \phi_2$ difference

in coordinates (x', y') rotated to align with *principal axes*

$$E'_x = \mathcal{E}_0 \ \cos\beta \ \cos(\omega t) \quad E'_y = -\mathcal{E}_0 \ \sin\beta \ \sin(\omega t)$$

for some $\beta \in [-\pi/2, +\pi/2]$
Q: evolution if $\beta > 0$?



$$E'_x = \mathcal{E}_0 \ \cos\beta \ \cos(\omega t) \qquad E'_y = -\mathcal{E}_0 \ \sin\beta \ \sin(\omega t)$$

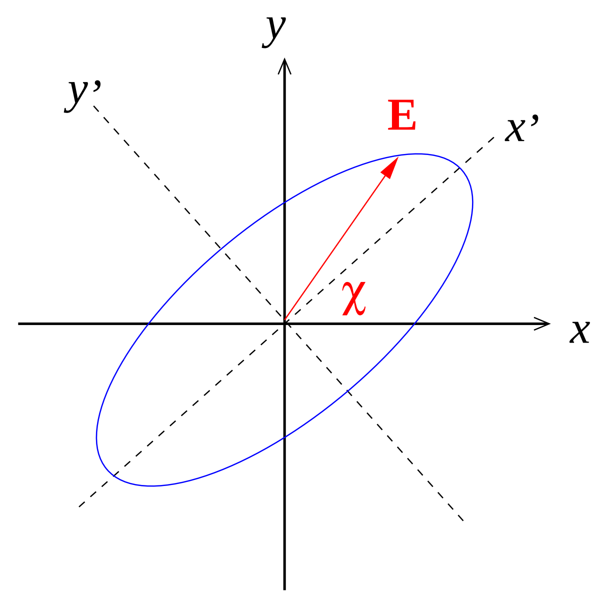
principle axes: $\mathcal{E}_0 \ \cos\beta$ and $\mathcal{E}_0 \ \sin\beta$

if $\beta \in [0, \pi/2]$: ellipse sweeps clockwise
$\rightarrow$ "*righthanded*" elliptical polarization, *negative helicity*
if $\beta \in [-\pi/2, 0]$: *"lefthanded", positive helicity*

Q: what β(s) give complete linear polarization? circular?

we want to relate $x - y$ **field parameters** $\mathcal{E}_1, \mathcal{E}_2, \phi_1, \phi_2$

to $x' - y'$ **principle axes parameters** $\mathcal{E}_0, \beta, \chi$



rotate $x - y$ components by angle χ

$$E_x = \mathcal{E}_0 \left(\cos\beta \cos\chi \cos\omega t + \sin\beta \sin\chi \sin\omega t \right)$$
$$E_y = \mathcal{E}_0 \left(\cos\beta \sin\chi \cos\omega t - \sin\beta \cos\chi \sin\omega t \right)$$

matching to, e.g., $E_x = \mathcal{E}_1 \cos(\omega t - \phi_1)$:

$$\mathcal{E}_1 \cos\phi_1 = \mathcal{E}_0 \cos\beta \cos\chi \tag{3}$$
$$\mathcal{E}_1 \sin\phi_1 = \mathcal{E}_0 \sin\beta \sin\chi \tag{4}$$
$$\mathcal{E}_2 \cos\phi_2 = \mathcal{E}_0 \cos\beta \sin\chi \tag{5}$$
$$\mathcal{E}_2 \sin\phi_2 = -\mathcal{E}_0 \sin\beta \cos\chi \tag{6}$$

Q: what causes polarization in the first place?

Preview: What Causes Polarization?

polarization is a vector: indicates a preferred direction
source needs to have special orientation

Magnetic Fields

magnetic fields encode special direction in particle motion
emission reflects this

Scattering

scattering introduces special direction: incident radiation
polarization varies relative to this

`www: Awesome examples: blue sky, HL Tau, Orion, CMB`

Q: how can we determine polarization by intensity measurements with polarimeters?

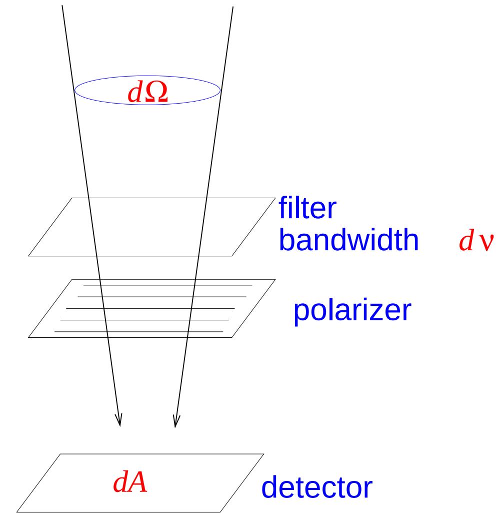
Introduce *polarizer*

can *rotate* polarizer:

$\rightarrow$ measure I_x, I_y, and 45° rotated $I_{x'}, I_{y'}$

can use circular polarizers (quarter wave plate) to measure

$\rightarrow$ positive and negative circular polarization I_+, I_-



combine: **Stokes parameters**

$$
\begin{aligned}
I &= I_x + I_y &(7)\\
Q &= I_x - I_y &(8)\\
U &= I_{x'} - I_{y'} &(9)\\
V &= I_+ - I_- &(10)
\end{aligned}
$$

Q: what physically is each? can more than one of Q, U, V be
∞ *nonzero? what does that correspond to?*

Q: range of values for Q? U? V? are they all independent?

Stokes Parameters

for *monochromatic waves*, Stokes parameters related to $\mathcal{E}_1, \mathcal{E}_2, \phi_1, \phi_2$ and $\mathcal{E}_0, \beta, \chi$ bases:

$$
\begin{aligned}
I &= \mathcal{E}_1^2 + \mathcal{E}_2^2 = \mathcal{E}_0^2 && (11)\\
Q &= \mathcal{E}_1^2 - \mathcal{E}_2^2 = \mathcal{E}_0^2 \cos 2\beta \cos 2\chi && (12)\\
U &= 2\mathcal{E}_1\mathcal{E}_2 \cos(\phi_1 - \phi_2) = \mathcal{E}_0^2 \cos 2\beta \sin 2\chi && (13)\\
V &= 2\mathcal{E}_1\mathcal{E}_2 \sin(\phi_1 - \phi_2) = \mathcal{E}_0^2 \sin 2\beta && (14)
\end{aligned}
$$

and thus

$$
\begin{aligned}
\mathcal{E}_0 &= \sqrt{I} && (15)\\
\sin 2\beta &= V/I && (16)\\
\tan 2\chi &= U/Q && (17)
\end{aligned}
$$

since wave has 3 independent parameters,
Stokes parameters must be *related*

$$I^2 = Q^2 + U^2 + V^2 \tag{18}$$

Quasi-Monochromatic Waves

natural light generally not a pure monochromatic wave
with a single, definite, complete state of polarization

rather: a *superposition* of components with many polarizations

consider wave with *slowly varying* amplitudes and phases

$$E_1(t) = \mathcal{E}_1(t)\ e^{i\phi_1(t)}\ ; \quad E_2(t) = \mathcal{E}_2(t)\ e^{i\phi_2(t)} \tag{19}$$

"slow": wave looks completely polarized on timescalse ω^{-1}
but amplitudes and phases drift over intervals $\Delta t \gg \omega^{-1}$
→ polarization changes

but also wave is *no longer monochromatic*
frequency spread: "*bandwidth*" $\Delta\omega \sim 1/\Delta t \ll \omega$
→ *quasi-monochromatic wave*

Q: effect on Stokes?

Stokes Parameters for Quasi-Monochromatic Light

real measurements represent averages over timescales during which polarization can change

Stokes parameters become averages

$$\begin{aligned} I &= \langle E_1 E_1^* \rangle + \langle E_2 E_2^* \rangle = \left\langle \mathcal{E}_1^2 + \mathcal{E}_2^2 \right\rangle && (20) \\ Q &= \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle = \left\langle \mathcal{E}_1^2 - \mathcal{E}_2^2 \right\rangle && (21) \\ U &= \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle = 2 \langle \mathcal{E}_1 \mathcal{E}_2 \cos(\phi_1 - \phi_2) \rangle && (22) \\ V &= -i \left(\langle E_1 E_2^* \rangle - \langle E_2 E_1^* \rangle \right) = 2 \langle \mathcal{E}_1 \mathcal{E}_2 \sin(\phi_1 - \phi_2) \rangle && (23) \end{aligned}$$

but for quasi-monochromatic waves

$$I^2 \geq Q^2 + U^2 + V^2 \qquad (24)$$

- quasi-monochromatic polarization is still in general *elliptical*
- but drifts can reduce degree of polarization

$$I^2 \geq Q^2 + U^2 + V^2 \tag{25}$$

- maximum polarization when equality holds: *completely elliptically polarized*
- minimum when $Q = U = V = 0$: *unpolarized*
- arbitrary wave is *partially polarized*

Q: what if we source is polarized, but sky pattern varies on angular scales below resolution?

Note: if source is polarized (mabye even fully)
but polarization pattern varies on scales below angular resolution

- then resolution "beam" averages over the pattern
- polarization from misaligned regions will partially *cancel*
  "beam dilution" of polarizaiton
- observe partial polarization

useful to define *polarized* intensity

$$I_{\text{pol}} = Q^2 + U^2 + V^2 \tag{26}$$

and since $I_{\text{pol}} \leq I$, define fractional **degree of polarization**

$$\Pi \equiv \frac{I_{\text{pol}}}{I} = \frac{\sqrt{Q^2 + U^2 + V^2}}{I} \tag{27}$$

note: can always decompose Stokes parameters

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} I - I_{\rm pol} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} I_{\rm pol} \\ Q \\ U \\ V \end{pmatrix} \tag{28}$$

sum of unpolarized and polarized components

Superposition and Stokes

consider composite wave that is superposition of many independent waves
electric field components are given by **superposition**

$$E_1 = \sum_k E_1^{(k)} \quad ; \quad E_2 = \sum_k E_1^{(k)} \tag{29}$$

each term k of which has different phase

PS4: phases specified, can calculate sum explicitly

but generally, *phases are random*
so field products average out phases from different waves

$$\left\langle E_i E_j^* \right\rangle = \sum_k \sum_\ell \left\langle E_i^{(k)} E_j^{(\ell)*} \right\rangle = \sum_k \left\langle E_i^{(k)} E_i^{(k)*} \right\rangle \tag{30}$$

but due to this averaging, *Stokes parameters are additive*

$$I = \sum_k I^{(k)} \quad (31)$$

$$Q = \sum_k Q^{(k)} \quad (32)$$

$$U = \sum_k U^{(k)} \quad (33)$$

$$V = \sum_k V^{(k)} \quad (34)$$

How Do Charges Generate Radiation

Thus far: **vacuum** Maxwell solutions support EM waves

- speed c
- transverse
- $\vec{B} = \vec{n} \times \vec{E}$

Maxwell **sources** are charges and currents

But how do sources *generate* radiation?
Strategy: study point charge, then superpose

Consider a point charge *at rest*
Q: what are ρ, $\vec{j}$ *everywhere?* $\vec{E}$, $\vec{B}$ *everywhere?*

A Point Charge at Rest

Consider a point charge q at rest at origin $\vec{r} = 0$
charge density $\rho = q\delta(\vec{r})$
current density $\vec{j} = \rho\vec{v} = 0$

Gauss' Law: $\nabla \cdot \vec{E} = 4\pi\,\rho$
Spherical symmetry: $\vec{E} = E(r)\ \hat{r}$
Gauss' Theorem applied to sphere enclosing charge:

$$\int \nabla \cdot \vec{E}\ dV = \int \vec{E} \cdot d\vec{A} \int E\ dA = 4\pi r^2\ E \qquad (35)$$

$$= 4\pi \int \rho\ dV = 4\pi\ q \qquad (36)$$

$$E(r) = \frac{q}{r^2} \qquad (37)$$

Coulomb's Law!

and $\vec{j} = 0$ means $\vec{B} = 0$: no magnetic field

Q: how can things change if the charge moves?

An Accelerated Point Charge

consider a particle rapidly *deceleraled* from speed v to rest over time δt

consider a later time $t \gg \delta t$

Q: field configuration near *particle* ($r \ll ct$) *?*

Q: field configuration near *particle* ($r \gg ct$)*?*

Q: consequences?

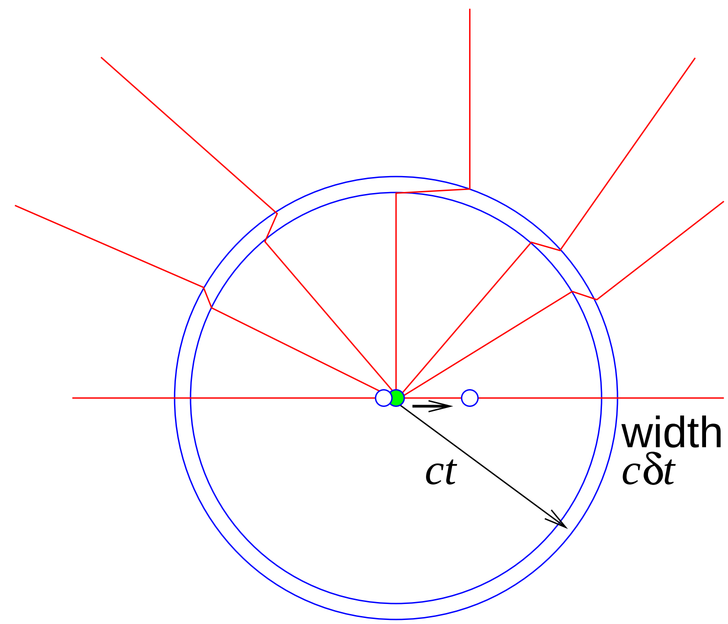
for fields track particle location expected for constant velocity

- nearby: $r \ll ct$, fields radial around particle at rest
- far away: $r \gg ct$: fields don't "know" particle has stopped
  $\rightarrow$ "anticipate" location displaced by ct from original particle
  radially oriented around this expected point

between the two regimes: $r = ct \pm c\delta t$

field lines must have "kinks" which

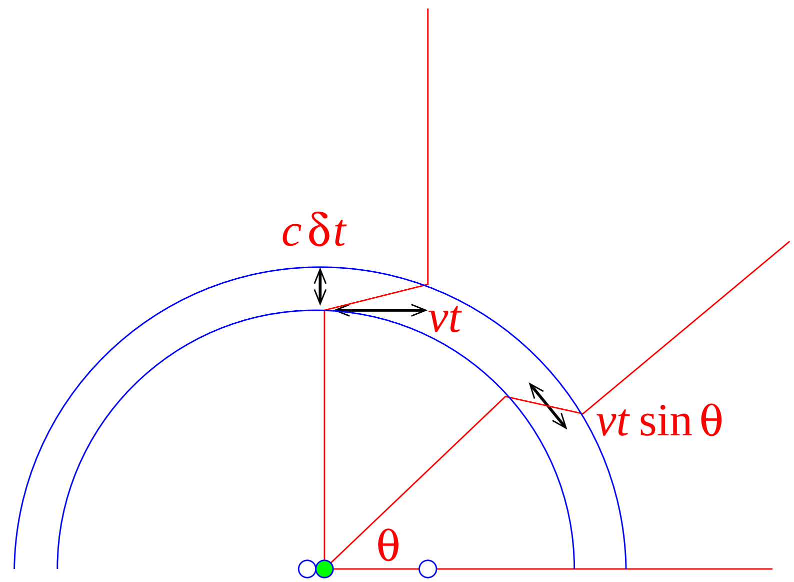
- have tangential field component
- tangential component is *anisotropic*
  and largest $\perp \vec{v}$



consider *vertical fieldline* $\perp \vec{v}$:

kink radial width $c\delta t$

kink tangential width $vt = (v/c)r$



tangential/radial ratio is $(v/\delta t)r/c^2$

but $v/\delta t = a$, average acceleration:

$\rightarrow E_\perp/E_r = ar/c^2$

more generally, tangential width is

$vt \sin\Theta = (v/c)r \sin\Theta$

and so using Coulomb for E_r:

$$E_\perp = \frac{ar\sin\Theta}{c^2} E_r = \frac{qa}{c^2 r}\sin\Theta \quad (38)$$

this is huge! *Q: why?*

Q: relation to radiated flux?

We find acceleration leads to a propagating field perturbation that is **tangential = transverse!**
just what we expect for EM radiation

so we expect also a transverse $\vec{B}$ component, with

$$B_\perp = E_\perp = \frac{ar \sin \Theta}{c^2} E_r = \frac{qa}{c^2 r} \sin \Theta \tag{39}$$

and thus a radial Poynting vector with magnitude

$$S = \frac{c}{4\pi} E_\perp^2 = \frac{q^2 a^2}{4\pi c^3 r^2} \sin^2 \Theta \tag{40}$$

this is also huge! *Q: why?*

Q: total radiated power per solid angle?

Larmor Formula

Poynting flux:

$$S = \frac{c}{4\pi} E_{\perp}^2 = \frac{q^2 a^2}{4\pi c^3 r^2} \sin^2 \Theta \tag{41}$$

- scales as $S \propto 1/r^2$! as it must!
- note importance of $E_{\perp} \propto 1/r$ scaling

Total power into solid angle $d\Omega$: $dP = r^2 \ S \ d\Omega$
so power per solid angle

$$\frac{dP}{d\Omega} = r^2 S = \frac{c r^2 E_{\perp}^2}{4\pi} = \frac{q^2 a^2}{4\pi c^3} \sin^2 \Theta \tag{42}$$

Larmor Formula for Radiated power

Q: lessons from magnitude? direction?

Larmor:

$$\frac{dP}{d\Omega} = r^2 S = \frac{c r^2 E_\perp^2}{4\pi} = \frac{q^2 a^2}{4\pi c^3} \sin^2\Theta \tag{43}$$

- magnitude $P \propto a^2$: **accelerated charges radiate**
- direction: $dP/d\Omega \propto \sin^2\Theta$
  not isotropic!
  maximum orthogonal to acceleration
  zero along acceleration