Astronomy 501: Radiative Processes

Lecture 14

Sept 23, 2022

Announcements:

- **Problem Set 4 extended to Sunday**
- **Problem Set 5 out today, due next Friday**

Last time: finished polarization

Today: radiation from moving charges

How Do Charges Generate Radiation

Thus far: **vacuum** Maxwell solutions support EM waves

- speed c
- transverse
- $\vec{B} = \vec{n} \times \vec{E}$

Maxwell **sources** are charges and currents

But how do sources *generate* radiation?
Strategy: study point charge, then superpose

Consider a point charge *at rest*
Q: what is ρ *everywhere?* $\vec{E}$ *everywhere?*

A Point Charge at Rest

Consider a point charge q at rest at origin $\vec{r} = 0$
charge density $\rho = q\ \delta(\vec{r})$

Gauss' Law: $\nabla \cdot \vec{E} = 4\pi\,\rho$
Spherical symmetry: $\vec{E} = E(r)\ \hat{r}$

- field lines point back to charge
- field lines are isotropic

Gauss' Theorem applied to sphere enclosing charge:

$$\begin{aligned}
\int \nabla \cdot \vec{E}\ dV &= \int \vec{E} \cdot d\vec{A} \int E\ dA = 4\pi r^2\ E && (1) \\
&= 4\pi \int \rho\ dV = 4\pi\ q && (2) \\
E(r) &= \frac{q}{r^2} && (3)
\end{aligned}$$

Coulomb's Law! Static charge field $E \propto 1/r^2$

Q: what is current density $\vec{j}$ everywhere? $\vec{B}$ everywhere?

Point Charge at Rest: Magnetic Fields

current density $\vec{j} = \rho \vec{v} = 0$
because $\vec{v} = 0$ at rest

and $\vec{j} = 0$ means $\vec{B} = 0$: **no magnetic field**

Q: how can things change if the charge moves?

Moving Charges: No Acceleration

If a charge moves relative to observer
with *constant velocity* $\vec{v}$
and thus *no acceleration*:

Proper treatment must include relativity

Preview of relativistic result:

- field direction still radial!
- points to present position of charge!
- also: field lines no longer isotropic
  lines denser (i.e., field strengths higher) when $\perp \vec{v}$

Legal? yes! velocity constant, trajectory always "available"

An Accelerated Point Charge

consider a particle rapidly *decelerated* from speed v to rest over time δt

consider a later time $t \gg \delta t$

Q: field configuration near *particle* ($r \ll ct$) *?*

Q: field configuration far from *particle* ($r \gg ct$)*?*

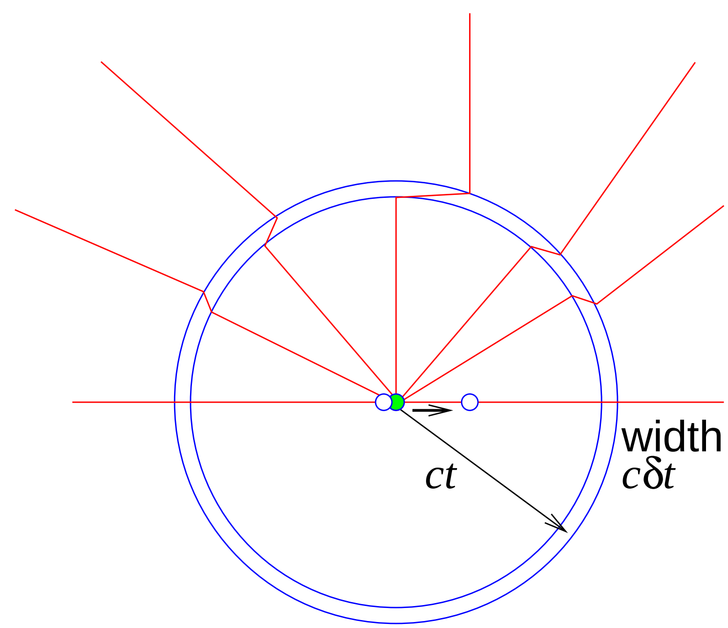
Q: consequences?

for fields track particle location expected for constant velocity

- nearby: $r \ll ct$, fields radial around particle at rest
- far away: $r \gg ct$: fields don't "know" particle has stopped
  $\rightarrow$ "anticipate" location displaced by ct from original particle
  radially oriented around this expected point

between the two regimes: $r = ct \pm c\delta t$
field lines must have "kinks"

- *Q: how fast does kink move?*
- *Q: what is kink direction?*
- *Q: what will a distant observer see?*
- *Q: why is this a Big Deal?*

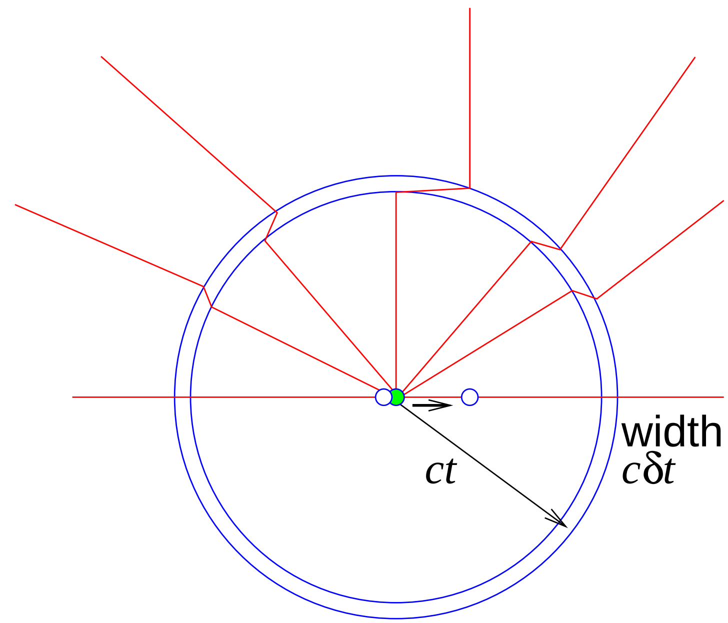


Getting the Kinks Out

between the two regimes: $r = ct \pm c\delta t$

field lines must have "kinks" which

- propagate at speed c
- have tangential field component
- tangential component is *anisotropic* and largest $\perp \vec{v}$



distant observer will see pulse

- propagating at c
- tangential to motion: transverse
- in direction $\perp v$: pulse amplitude larger than radial field

this is radiation! caused by acceleration!

consider *vertical fieldline* $\perp \vec{v}$:

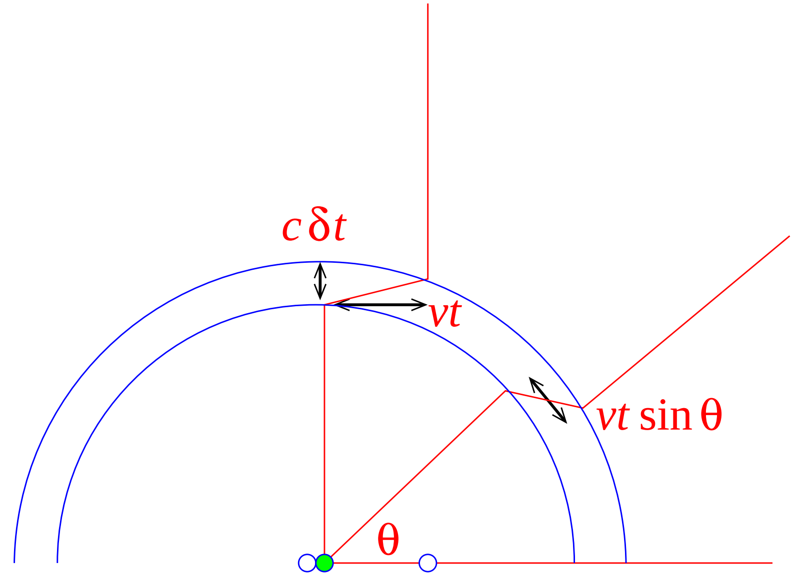
kink radial width $c\delta t$

kink tangential width $vt = (v/c)r$

tangential/radial ratio is $(v/\delta t)r/c^2$

but $v/\delta t = a$, average acceleration:

$\rightarrow E_\perp/E_r = ar/c^2$



more generally, tangential width is

$vt \sin\Theta = (v/c)r \sin\Theta$

with angle Θ *between* $\vec{a}$ *and* $\hat{n}$

and so using Coulomb for E_r:

$$E_\perp = \frac{ar\sin\Theta}{c^2} E_r = \frac{qa}{c^2 r}\sin\Theta \qquad (4)$$

this is huge! *Q: why?*

Q: relation to radiated flux?

We find acceleration leads to a propagating field perturbation that is **tangential = transverse!**
just what we expect for EM radiation

so we expect also a transverse $\vec{B}$ component, with

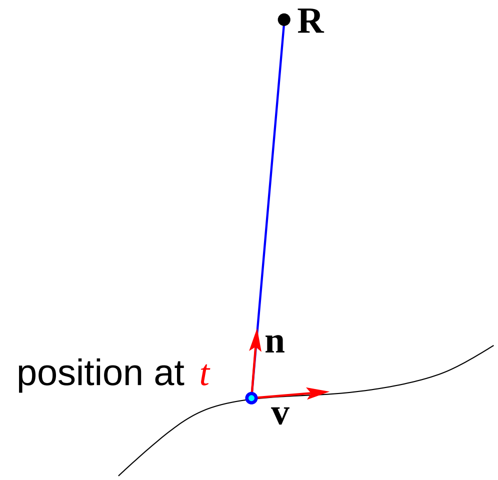
$$B_\perp = E_\perp = \frac{ar \sin \Theta}{c^2} E_r = \frac{qa}{c^2 r} \sin \Theta \tag{5}$$

and thus a radial Poynting vector with magnitude

$$S = \frac{c}{4\pi} E_\perp^2 = \frac{q^2 a^2}{4\pi c^3 r^2} \sin^2 \Theta \tag{6}$$

Electrodynamics of Moving Charges: Strategy

point charge q:
position $\vec{R}(t)$
velocity $\vec{v} = \dot{\vec{R}} = \vec{\beta} c$
and acceleration $\vec{a} = \ddot{\vec{R}} = c\ d\vec{\beta}/dt = c\dot{\vec{\beta}}$



Maxwell sources:
charge density $\rho(\vec{x}) = q\ \delta(\vec{x} - \vec{R})$, current density $\vec{j} = \rho\vec{v}$

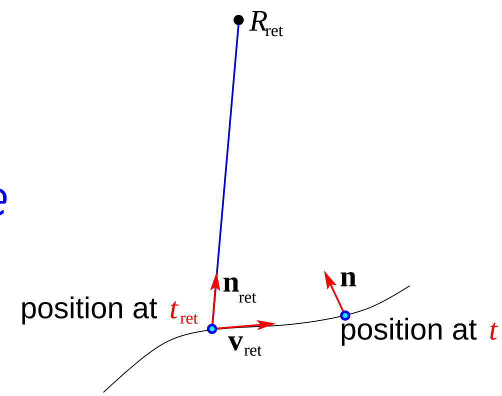
Procedure (see R&L and Extras for more):
0. Use full Special Relativity
1. write EM fields as derivatives of *4-potential* $(\phi, \vec{A})$
2. *Maxwell* $\rightarrow$ 2nd-order equations $\partial^2\text{potential} = \text{source}$
3. solve for fields given above source terms

Electrodynamics of Moving Charges: Results

A careful calculation, and a lot of algebra, gives
an exact formula for the field of a moving point charge

$$\vec{E}(\vec{R},t) = q\left[\frac{(\hat{n}-\vec{\beta})(1-\beta^2)}{\kappa^3 R^2}\right]_{\rm ret} + \frac{q}{c}\left[\frac{\hat{n}}{\kappa^3 R} \times \left\{(\hat{n}-\vec{\beta}) \ \times \dot{\vec{\beta}}\right\}\right]_{\rm ret}$$

where $\kappa = 1 - \hat{n}\cdot\hat{\beta}$
and "ret" = particle position at *retarded time*
$t_{\rm ret} = t - R/c$



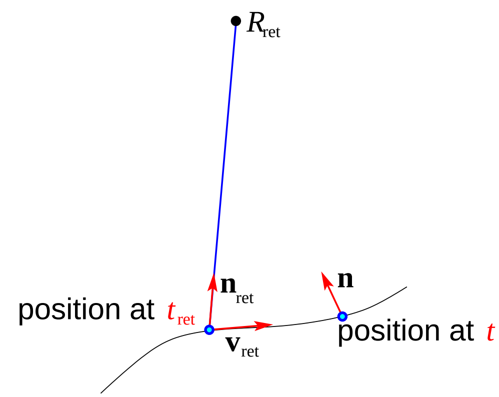
form is rich = complicated, but also complete and exact!
depends on charge position, velocity, and acceleration

for electric field

$$\vec{E}(\vec{R},t) = q\left[\frac{(\hat{n}-\vec{\beta})(1-\beta^2)}{\kappa^3 R^2}\right]_{\text{ret}} + \frac{q}{c}\left[\frac{\hat{n}}{\kappa^3 R}\times\left\{(\hat{n}-\vec{\beta}) \;\; \times \dot{\vec{\beta}}\right\}\right]_{\text{ret}}$$

with $\kappa = 1 - \hat{n}\cdot\hat{\beta}$

magnetic field is $$\vec{B}(\vec{R},t) = \left[\hat{n}\times\vec{E}(\vec{R},t)\right]_{\text{ret}}$$



Q: $\vec{E}$ result for charge at rest? $\vec{B}$?

Q: $\vec{E}$ for charge with with constant velocity?

Q: result at large R?

Electric "Velocity" Field

point source first term = "velocity field"

$$\vec{E}(\vec{R},t)_{\text{vel}} = q\left[\frac{(\hat{n}-\vec{\beta})(1-\beta^2)}{\kappa^2 R^2}\right]_{\text{ret}} \tag{7}$$

- depends only on position and velocity evaluated at a *past* location of the particle
- velocity field *not isotropic* if particle moving

displacement from retarded position $\vec{R}(t_{\text{ret}})$
to the field position $\vec{R}$ is $\hat{n}c(t - t_{\text{ret}})$
to the current particle position $\beta c(t - t_{\text{ret}})$
so $\vec{E}$ *points to current* position!
→ legal? yes! velocity constant, trajectory always "available"

Electric Acceleration Field

electric velocity field $\propto 1/R^2$
but other *acceleration* term $\propto \dot{v}_0$

$$\vec{E}(\vec{R},t)_{\text{accel}} = \frac{q}{c}\left[\frac{\hat{n}}{\kappa^3 R} \times \left\{(\hat{n} - \hat{\beta}) \ \times \dot{\hat{\beta}}\right\}\right]_{\text{ret}} \quad (8)$$

drops with distance $\propto 1/R$: always larger at large R

for nonrelativistic motion, $\beta_0 = v_0/c \ll 1$,
and so to first order

$$\vec{E}(\vec{R},t)_{\text{accel}} \approx \left[\frac{q}{c^2 R}\hat{n} \times (\hat{n} \times \vec{a})\right]_{\text{ret}} \quad (9)$$

a huge result!

Q: if acceleration is linear, what is polarization?

at large distances

$$\vec{E}(\vec{R},t) \to \vec{E}(\vec{R},t)_{\text{accel}} \approx \left[\frac{q}{c^2 R}\hat{n} \times (\hat{n} \times \vec{a})\right]_{\text{ret}} \tag{10}$$

instantaneous $\vec{E}$ *direction* set by $\hat{a}$ and $\hat{n}$

if acceleration is linear $\to$ $\hat{a}$ fixed
then $\vec{E}$ lies within $(\hat{n}, \hat{a})$ plane $\to$ *100% linearly polarized*

using $\vec{B} \to \hat{n} \times \vec{E}_{\text{accel}}$, the Poynting flux is

$$\vec{S} \approx \frac{c}{4\pi} E^2_{\text{accel}}\ \hat{n} = \frac{q^2}{4\pi c^3 R^2}\left|\hat{n} \times (\hat{n} \times \dot{\vec{\beta}})\right|^2 \hat{n} \tag{11}$$

Q: noteworthy features?

the Poynting flux is

$$\vec{S} \approx \frac{q^2}{4\pi c^3 R^2} \left| \hat{n} \times (\hat{n} \times \dot{\vec{\beta}}) \right|^2 \tag{12}$$

$S \propto R_{\text{ret}}^{-2}$: *flux obeys inverse square law!*

Power per unit solid angle is

$$\frac{dP}{d\Omega} = R^2 \hat{n} \cdot \vec{S} \approx \frac{c}{4\pi} |R\vec{E}_{\text{accel}}|^2 = \frac{q^2}{4\pi c^3} \left| \hat{n} \times (\hat{n} \times \dot{\vec{\beta}}) \right|^2 \tag{13}$$

independent of distance! *Q: why did this have to be true?*

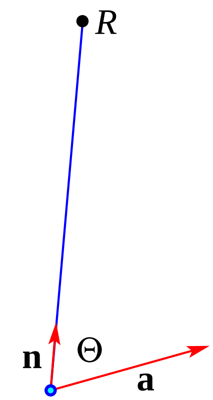
Q: in which directions is $dP/d\Omega$ *largest? smallest?*
Q: radiation pattern?

Larmor Formula

Nonrelativistic charges radiate when accelerated!

Power per unit solid angle is

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} \left| \hat{n} \times (\hat{n} \times \dot{\vec{\beta}}) \right|^2$$



define angle Θ *between* $\vec{a}$ *and* $\hat{n}$ via $\hat{n} \cdot \hat{\beta} = \cos\Theta$:

$$\frac{dP}{d\Omega} = \frac{q^2 a^2}{4\pi c^3} \sin^2\Theta$$

a $\sin^2\Theta$ pattern!

$\rightarrow$ no radiation in direction of acceleration, maximum $\perp \vec{a}$
integrate over all solid angles: *total radiated power* is

$$P = \frac{q^2 a^2}{4\pi c^3} \int \sin^2\Theta \, d\Omega = \frac{2}{3} \frac{q^2}{c^3} a^2 \tag{14}$$

this will be our workhorse!

relates radiation to particle acceleration via $P \propto a^2$

Director's Cut Extras

The Vector Potential

No-monopoles condition $\nabla \cdot \vec{B}$
strongly restricts $\vec{B}$ configurations

condition *automatically* satisfied if we write

$$\vec{B} = \nabla \times \vec{A} \tag{15}$$

guarantees zero divergence because, for *any* $\vec{A}$

$$\nabla \cdot (\nabla \times \vec{A}) = 0 \tag{16}$$

where $\vec{A}$ is the **vector potential**
Q: units of $\vec{A}$?

write Faraday's law in terms of $\vec{A}$:

$$\nabla \times \vec{E} = -\frac{1}{c}\partial_t(\nabla \times \vec{A}) \tag{17}$$

and so

$$\nabla \times \left(\vec{E} + \frac{1}{c}\partial_t\vec{A}\right) = 0 \tag{18}$$

strongly restricts $\vec{E}$ configurations

Q: how to automatically satisfy?

The Scalar Potential

Faraday with $\vec{A}$

$$\nabla \times \left(\vec{E} + \frac{1}{c} \partial_t \vec{A} \right) = 0 \tag{19}$$

vector field $\vec{E} + \frac{1}{c}\partial_t \vec{A}$ is curl-free

to automatically satisfy this, note that

$$\nabla \times (\nabla \phi) = 0 \tag{20}$$

curl of grad vanishes for any scalar field (=function) ϕ

define **scalar potential** via

$$\vec{E} = -\nabla \phi - \frac{1}{c} \partial_t \vec{A} \tag{21}$$

Q: units of ϕ?

Q: are $\vec{A}$ and ϕ unique? why?

Gauge Freedom

vector potential defined to give $\nabla \times \vec{A} = \vec{B}$
clearly if $\vec{A} \to \vec{A}' = \vec{A} + \text{constant}$, $\vec{B} \to \vec{B}$
$\Rightarrow$ physical field unchanged

in fact: $\vec{B}$ unchanged for *any transformation*
$\vec{A} \to \vec{A}'$ which preserves $\nabla \times \vec{A}' = \vec{B}$:

$$\nabla \times (\vec{A}' - \vec{A}) = 0 \tag{22}$$

and thus there is no physical change if

$$\vec{A}' = \vec{A} + \nabla\psi \tag{23}$$

because $\nabla \times (\nabla\psi) = 0$ for any ψ
$\to$ *gauge invariance*

Q: what condition needed to keep $\vec{E}$ unchanged?

Gauge Invariance

the physical electric field has

$$\vec{E} = -\nabla\phi - \frac{1}{c}\partial_t\vec{A} \tag{24}$$

and must remain the same when $\vec{A} \to \vec{A} + \nabla\psi$

but we have

$$\vec{E} \to \vec{E}' = -\nabla\phi - \frac{1}{c}\partial_t\vec{A}' \tag{25}$$

$$= -\nabla\left(\phi + \frac{1}{c}\partial_t\psi\right) - \frac{1}{c}\partial_t\vec{A} \tag{26}$$

Q: and so?

$$\vec{E} \to \vec{E}' \;=\; -\nabla\left(\phi + \frac{1}{c}\partial_t\psi\right) - \frac{1}{c}\partial_t\vec{A} \tag{27}$$

and so to keep $\vec{E}' = \vec{E}$ requires

$$\phi \to \phi' = \phi - \frac{1}{c}\partial_t\psi \tag{28}$$

the $\vec{E}, \vec{B}$ preserving mappings

$$(\phi, \vec{A}) \to (\phi, \vec{A}) + (\partial_t\psi/c, \nabla\psi) \tag{29}$$

is a **gauge transformation**

a deep but also annoying property of electromagnetism
for our purposes, a useful but not unique choice

$$\nabla \cdot \vec{A} + \frac{1}{c}\partial_t\phi = 0 \tag{30}$$

"Lorentz gauge"

Maxwell Revisited

express Maxwell in terms of potentials: Coulomb

$$-\nabla \cdot \left(\nabla\phi - \frac{1}{c}\partial_t \vec{A}\right) = -\nabla^2\phi - \frac{1}{c}\partial_t(\nabla \cdot \vec{A}) \quad (31)$$

$$= 4\pi\rho_q \quad (32)$$

and so in Lorentz gauge

$$\nabla^2\phi - \frac{1}{c^2}\partial_t^2\phi = -4\pi\rho_q \quad (33)$$

scalar potential satisfies a wave equation!
ϕ source is charge density ρ_q
changes in ϕ propagate at speed c

for *static* situation $\partial_t\phi = 0$, Poisson $\nabla^2\phi = -4\pi\rho_q$, and

$$\phi(\vec{r}) = \int d^3\vec{r}' \; \frac{\rho_q(\vec{r}')}{|\vec{r}' - \vec{r}|} \quad (34)$$

Q: solution for full wave equation?

Scalar Potential and Retarded Time

general solution to

$$\nabla^2 \phi - \frac{1}{c^2} \partial_t^2 \phi = -4\pi \rho_q \tag{35}$$

turns out to be

$$\phi(\vec{r}, t) = \int d^3\vec{r}' \ \frac{\rho_q(\vec{r}', t')}{|\vec{r}' - \vec{r}|} = \int d^3\vec{r}' \ \left[\frac{\rho_q}{|\vec{r}' - \vec{r}|} \right]_{\text{ret}} \tag{36}$$

where source density $\rho_q(\vec{r}', t')$

is evaluated at **retarded time**

$$t' \equiv [t_{\text{ret}}] = t - \frac{|\vec{r} - \vec{r}'|}{c} \tag{37}$$

$\rightarrow \phi$ "learns" about changes in charge density at $\vec{r}'$

only after signal propagation time $ct_{\text{prop}} = |\vec{r}'|$

Maxwell and the Vector Potential

in terms of potentials, Ampère in Cartesian coords:

$$\nabla \times (\nabla \times \vec{A}) = \nabla^2 \vec{A} - \nabla(\nabla \cdot \vec{A}) \tag{38}$$

$$= \frac{4\pi}{c}\vec{j} + \frac{1}{c}\left(\nabla\phi + \partial_t \vec{A}\right) \tag{39}$$

so in Lorentz gauge

$$\nabla^2 \vec{A} - \frac{1}{c^2}\partial_t^2 \vec{A} = -\frac{4\pi}{c}\vec{j} \tag{40}$$

vector potential also satisfies a wave equation
source is current density $\vec{j}$

Q: solution?

each component A_i of vector potential satisfies

$$\nabla^2 A_i - \frac{1}{c^2}\partial_t^2 A_i = -\frac{4\pi}{c} j_i \tag{41}$$

formally identical to scalar potential equation
if we put $\phi \to A_i$ and $\rho_q \to j_i/c$

and thus we can import the solution:

$$A_i(\vec{r},t) = \int d^3\vec{r}' \ \left[\frac{j_i}{|\vec{r}' - \vec{r}|}\right]_{\text{ret}} \tag{42}$$

$\to$ vector potential responds to current changes
after "retarded time" delay

Integral solutions for ϕ and $\vec{A}$ are huge!
Q: why? what's the Big Deal?

Recipe for Electromagnetic Fields

our mission: find $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$
given charge $\rho_q(\vec{r},t)$ and current $\vec{j}(\vec{r},t)$ distributions

solution: first find potentials via

$$\phi(\vec{r},t) = \int d^3\vec{r}' \left[\frac{\rho_q}{|\vec{r}' - \vec{r}|}\right]_{\rm ret} \quad (43)$$

$$\vec{A}(\vec{r},t) = \int d^3\vec{r}' \left[\vec{j}|\vec{r}' - \vec{r}|\right]_{\rm ret} \quad (44)$$

from these, find fields via

$$\vec{E} = -\nabla\phi - \frac{1}{c}\partial_t\vec{A} \quad (45)$$

$$\vec{B} = \nabla \times \vec{A} \quad (46)$$

ta da!

in the 3-D spatial integrals

$$\phi(\vec{r},t) = -\int d^3\vec{r}' \left[\frac{\rho_q}{|\vec{r}' - \vec{r}|}\right]_{\text{ret}} \tag{47}$$

it is convenient (and pretty!) to recast as integrals over 4-D spacetime:

$$\phi(\vec{r},t) = -\int d^3\vec{r}'\; dt'\; \frac{\rho_q(\vec{r}',t')}{|\vec{r}' - \vec{r}|}\; \delta(t' - t + |\vec{r} - \vec{r}'|/c) \tag{48}$$

were the δ function enforces the retarded time condition

Q: What if charges are all pointlike?

Potentials from Point Charges

if N point charges, where ith charge q_i has trajectory with position $\vec{r}_i(t)$, and velocity $\vec{v}_i(t)$, then

$$\rho_q(\vec{r},t) = \sum_i q_i \ \delta^{(3)}(\vec{r}-\vec{r}_i) \tag{49}$$

$$\vec{j}(\vec{r},t) = \sum_i q_i \ v_i(t) \ \delta^{(3)}(\vec{r}-\vec{r}_i) \tag{50}$$

with Dirac δ-functions $\delta^{(3)}(\vec{r}-\vec{r}_i) = \delta(x-x_i)\ \delta(y-y_i)\ \delta(z-z_i)$

scalar potential due to *one charge* with $q_0, \vec{r}_0(t), \vec{v}_0(t)$ is

$$\phi(\vec{r},t) = q_0 \int d^3\vec{r}' \ dt' \ \frac{\delta^{(3)}(\vec{r}'-\vec{r}_0(t))}{|\vec{r}'-\vec{r}|} \ \delta(t'-t+|\vec{r}-\vec{r}'|/c) \tag{51}$$

space part of integral is easy

$$\phi(\vec{r},t) = q_0 \int dt' \ \frac{\delta\left(t'-t+|\vec{r}-\vec{r}_0(t')|/c\right)}{|\vec{r}-\vec{r}_0(t')|} \tag{52}$$

writing $\vec{R}(t') \equiv \vec{r} - \vec{r}_0(t')$
and $R(t') = |\vec{R}(t')|$, we have

$$\phi(\vec{r}, t) = q_0 \int dt' \, \frac{\delta\left(t' - t + R(t')/c\right)}{R(t)} \tag{53}$$

and now the final δ function is nontrivial

math aside: fun properties of the δ function
$\delta(x)$ designed to give

$$\int f(y) \; \delta(y - x) \; dy = f(x) \tag{54}$$

but if δ argument is a function of the integration variable

$$\int f(y) \; \delta\left(g(x)\right) \; dy = \sum_{\text{roots} j} \frac{f(g(x_j))}{|dg/dx|_{x_j}} \tag{55}$$

where root x_j is the jth solution to $y - g(x) = 0$

here: define $t'' = t' - t + R(t')/c$
then $dt'' = dt' + \dot{R}(t')/c \; dt'$

Liénard-Wiechert Potentials

for point source with arbitrary trajectory, we have

$$\phi(\vec{r}, t) = \frac{1}{1 - \hat{n} \cdot \hat{\beta}_0(t_{\rm ret})} \frac{q_0}{R} \tag{56}$$

where $\hat{n} = \vec{r}/r$ and $\vec{\beta}_0(t) = \vec{v}_0(t)/c$

similarly, vector potential solution is

$$\vec{A}(\vec{r}, t) = \frac{1}{1 - \hat{r} \cdot \hat{\beta}_0(t_{\rm ret})} \frac{q_0 \vec{v}_0(\vec{r}, t_{\rm ret})}{R(t_{\rm ret})} \tag{57}$$

these are the **Liénard-Wiechert potentials**

Q: equipotential surfaces $\phi = const$ for stationary charge $\vec{r}_0(t) = const$?
Q: for charge with $\vec{v}_0$ large?
Q: implications?

potential factor $\kappa \equiv [1 - \hat{n} \cdot \hat{\beta}]_{\text{ret}}$ is

- directional,
- velocity dependent, such that
- *potential* $\propto 1/\kappa$ *enhanced along direction of charge motion* and *potential suppressed opposite direction of charge motion*

$\Rightarrow$ expect forward "beaming" effects!

But we want the EM fields, not just potentials,
so we need to evaluate

$$\vec{E} = -\nabla\phi - \frac{1}{c}\partial_t \vec{A} \tag{58}$$

$$\vec{B} = \nabla \times \vec{A} \tag{59}$$

using the beautiful Liénard-Wiechert point-source potentials
where, $\phi = \phi[\vec{r}, t; \vec{r}_0(t), \vec{v}_0(t)]$ and $\vec{A} = \vec{A}[\vec{r}, t; \vec{r}_0(t), \vec{v}_0(t)]$

Q: what terms will appear in $\vec{E}$*?*