Astronomy 501: Radiative Processes

Lecture 15

Sept 26, 2022

Announcements:

- **Problem Set 5 due Friday**

Last time: radiation from moving charges

- *Q: what motion leads to radiation?*

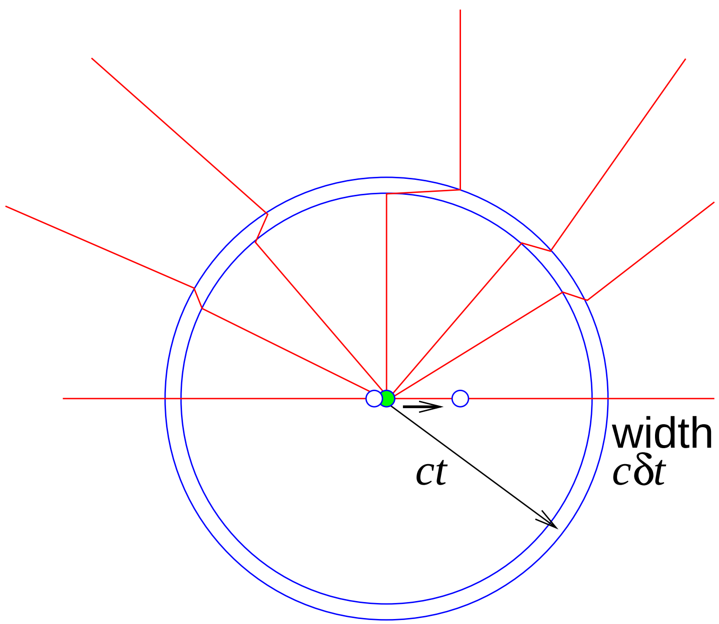
Getting the Kinks Out

decelerating charge

distant observer will see pulse

- propagating at c
- tangential to motion: transverse
- in direction $\perp v$: pulse amplitude larger than radial field

this is radiation! caused by acceleration!



for *non-relativistic* charge q with acceleration $\vec{a}$

viewed in direction $\hat{n}$

Q: acceleration field $\vec{E}$ dependence on R? direction? $\vec{B}$?

for *non-relativistic* charge q

with acceleration $\vec{a}$

viewed in at distance R and direction $\hat{n} = \vec{R}/R$

$$\vec{E}(\vec{R},t)_{\text{accel}} = \frac{q}{c}\left[\frac{\hat{n}}{\kappa^3 R} \times \left\{(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}\right\}\right]_{\text{ret}} \xrightarrow{\text{non-rel}} \frac{q}{c^2 R}\hat{n} \times (\hat{n} \times \vec{a})$$

and $\vec{B} = \hat{n} \times \vec{E}$

⋆ *field magnitude determines intensity*

⋆ *field direction determines polarization*

vector identity:

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \tag{1}$$

and so: $\hat{n} \times (\hat{n} \times \vec{a}) = (\hat{n} \cdot \vec{a})\hat{n} - \vec{a} = -[\vec{a} - (\hat{n} \cdot \vec{a})\hat{n}]$

Q: What is this physically? where does it point?

Non-Relativistic Acceleration: Polarization

field magnitude determines intensity
field direction determines polarization

$$\vec{E}_{\rm rad} \propto \hat{n} \times (\hat{n} \times \vec{a}) = -[\vec{a} - (\hat{n} \cdot \vec{a})\hat{n}] \equiv -\vec{a}_{\perp} \qquad (2)$$

where $\vec{a}_{\perp} = \vec{a} - (\hat{n} \cdot \vec{a})\hat{n}$

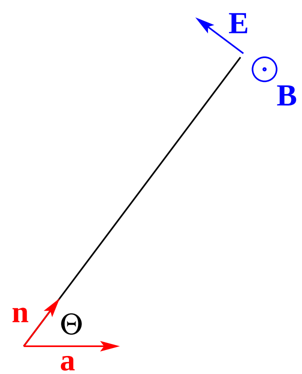
- $\hat{n} \cdot \vec{a}_{\perp} = 0$: orthogonal to $\hat{n}$
- $\hat{n} \cdot a$: component of $\vec{a}$ along $\hat{n}$

thus: $\vec{a}_{\perp}$ is accel component orthogonal to view direction $\hat{n}$

Lesson: $\vec{a}_{\perp}$ and hence *polarization direction* is

- along component of acceleration $\perp$ sightline
- the projection of $\vec{a}$ onto the observer's sky

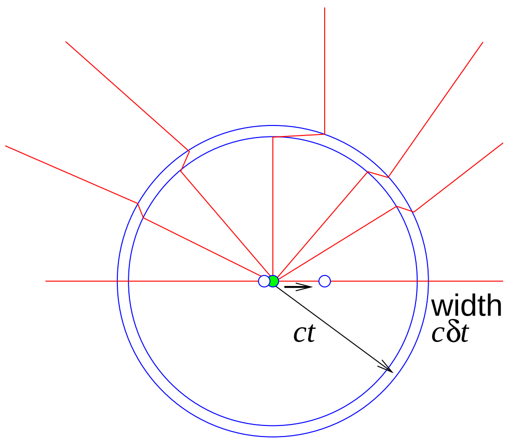
Q: polarization observed for linear acceleration?
Q: where maximum? where is pol and signal zero?



already saw example of linear acceleration

all observers see linear motion in projection

- radiation field 100% linearly polarized
- direction opposite acceleration
- rad field and intensity max $\perp \vec{a}$
- zero signal along $\vec{a}$
- in general: $E_{\text{rad}} \propto \sin \Theta$

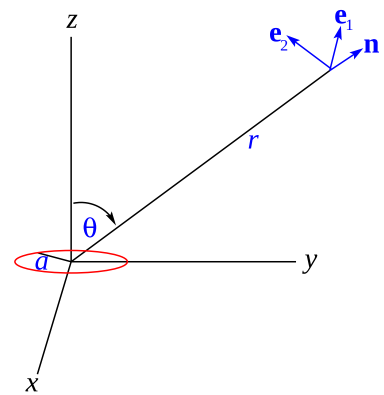


Now consider circular motion

Q: polarization observed along rotation axis?

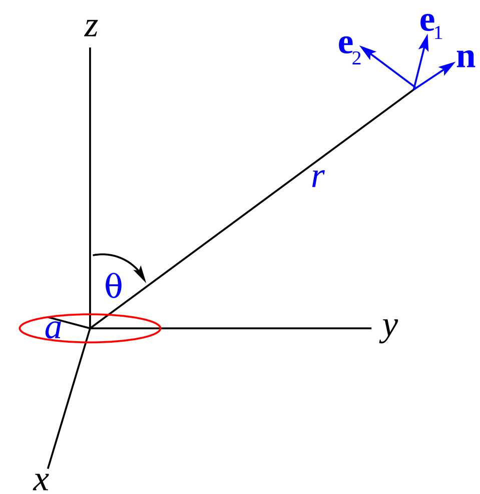
Q: polarization observed in plane of motion?

Q: polarization for arbitrary observer?



Nonrelativistic Uniform Circular Motion

For uniform circular motion:
observer sees projected orbit



- obs on orbit axis sees circular motion
  signal is 100% circularly polarized

- obs in plane of motion sees linear oscillation
  signal is 100% linearly polarized

- arbitrary observer sees elliptical orbit
  signal is 100% elliptically polarized

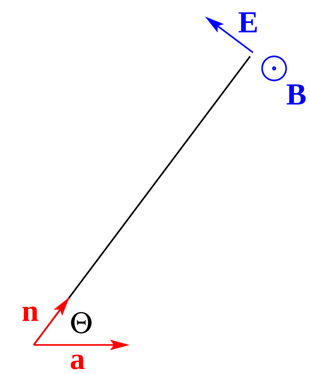
Q: What is polarization for helical (corkscrew) motion?

Larmor Formula

Nonrelativistic charges radiate when accelerated!

Power per unit solid angle is

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} \left| \hat{n} \times (\hat{n} \times \vec{a}) \right|^2$$



define angle Θ *between* $\vec{a}$ *and* $\hat{n}$ via $\hat{n} \cdot \hat{a} = \cos\Theta$:

$$\frac{dP}{d\Omega} = \frac{q^2 a^2}{4\pi c^3} \sin^2\Theta$$

a $\sin^2\Theta$ pattern!

$\rightarrow$ no radiation in direction of acceleration, maximum $\perp \vec{a}$
integrate over all solid angles: $P = q^2 a^2 / 4\pi c^3 \int \sin^2\Theta d\Omega$

total radiated power is

$$P = \frac{2}{3} \frac{q^2}{c^3} a^2$$

this will be our workhorse!
relates radiation to particle acceleration via $P \propto a^2$

An Ensemble of Point Charges

So far: field of a single point charge
Now: consider N particles, with q_i, $\vec{R}_i$, $\vec{v}_i = \dot{\vec{R}}_i, \vec{a}_i = \ddot{\vec{R}}_i$

Net $\vec{E}$ will be sum over all particles

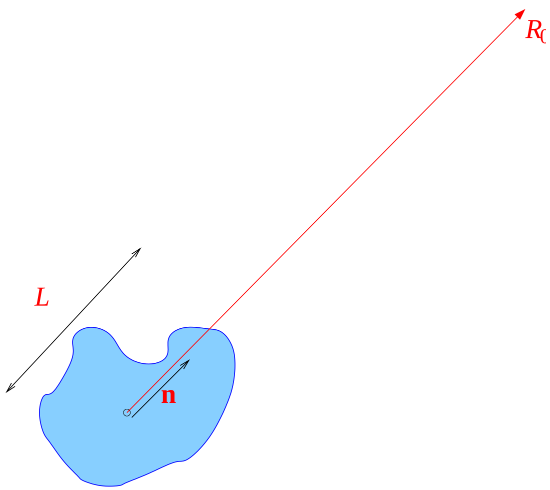
Q: relativisitic complications beyond "simple" bookkeeping?

Q: when will things simplify?

Approximate Phase Coherence

observed fields for each charge depend on its *retarded time*
and these are different for each charge
$\rightarrow$ leads to *phase differences* between particles
which we in general would have to track

When are phase differences not a problem?
When light-travel-time lags between particles
represent small phase differences



Let system size be L, and timescale for variations τ
if $\tau \gg L/c$, phase differences will be small

or: characteristic frequency is $\nu \sim 1/\tau$
so phase differences small if $c/\nu \gg L$, or $\lambda \gg L$
note that typical particle speeds $u \sim L/\tau$, so
phase coherence condition $\rightarrow u \ll c \rightarrow$ *nonrelativistic motion*

Dipole Approximation

so for **non-relativistic systems** we may ignore

- differences in time retardation, and
- the correction factor $\kappa = 1 - \hat{n} \cdot \vec{v}/c \rightarrow 1$

and thus we have

$$\vec{E}_{\text{rad}} = \sum_i \frac{q_i}{c^2} \frac{\hat{n} \times (\hat{n} \times \vec{a}_i)}{R_i} \tag{3}$$

but the system has $R_i \approx R_0 \gg L$, and so

$$\vec{E}_{\text{rad}} = \hat{n} \times \left(\frac{\hat{n}}{c^2 R_0} \times \sum_i q_i \vec{a}_i \right) = \frac{\hat{n} \times (\hat{n} \times \ddot{\vec{d}})}{c^2 R_0} \tag{4}$$

where the **dipole moment** is

$$\vec{d} = \sum_i q_i \vec{R}_i \tag{5}$$

for a non-relativistic dipole, we have

$$\vec{E}_{\rm rad} = \frac{\hat{n} \times (\hat{n} \times \ddot{\vec{d}})}{c^2 R_0} \tag{6}$$

this *dipole approximation* gives: power per unit solid angle

$$\frac{dP}{d\Omega} = \frac{\ddot{d}^{\,2}}{4\pi c^3} \sin^2 \Theta \tag{7}$$

and the total power radiated

$$P = \frac{2}{3} \frac{\ddot{d}^{\,2}}{c^3} \tag{8}$$

consider a dipole that maintains the same orientation $\vec{d}$

$$E(t) = \ddot{d}(t) \frac{\sin \Theta}{c^2 R_0} \tag{9}$$

using Fourier transform of $d(t)$, we have

$$d(t) = \int e^{-i\omega t} \tilde{d}(\omega) \; d\omega \tag{10}$$

and so

$$\tilde{E}(\omega) = -\omega^2 \tilde{d}(\omega) \frac{\sin \Theta}{c^2 R_0} \tag{11}$$

and thus the energy per solid angle and frequency is

$$\frac{dW}{d\Omega d\omega} = \frac{1}{c^3} \omega^4 \; \left| \tilde{d}(\omega) \right|^2 \sin^2 \Theta \tag{12}$$

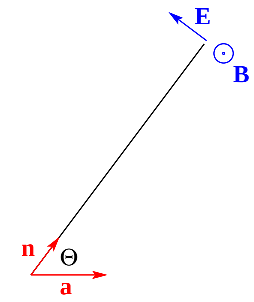
and

$$\frac{dW}{d\omega} = \frac{8\pi}{3c^3} \omega^4 \; \left| \tilde{d}(\omega) \right|^2 \tag{13}$$

- note the $\omega^4 \propto \lambda^{-4}$ dependence
- and $\tilde{d}(\omega)$: *dipole frequencies control radiation frequencies*

Radiation from Accelerated Charges: Polarization

Polarization is electric field direction $\vec{E}$
where $\vec{E} \perp \vec{B} \perp \hat{n}$



Observationally: use polarizer which selects out
one of two polarization states $\hat{\epsilon}_1, \hat{\epsilon}_2$
in some (complex) basis

e.g., if wave propagates in $\hat{n} = \hat{z}$ then

- xy polarization: $\epsilon_1 = \hat{x}$, $\epsilon_2 = \hat{y}$
- $x'y'$ polarizations: $\epsilon_1 = (\hat{x} + \hat{y})/\sqrt{2}$, $\epsilon_2 = (\hat{x} - \hat{y})/\sqrt{2}$
- circular polarization: $\epsilon_+ = (\hat{x} - i\hat{y})/\sqrt{2}$, $\epsilon_- = (\hat{x} + i\hat{y})/\sqrt{2}$

If (complex) electric vector is $\vec{E}$
Q: what passes through polarizer $\hat{\epsilon}_1$*?*

Complex electric vector is $\vec{E}$ can be written in some *polarization basis* $(\hat{\epsilon}_1, \hat{\epsilon}_2, \hat{n} = \hat{k})$ as

$$\vec{E} = (\mathcal{E}_1 \hat{\epsilon}_1 + \mathcal{E}_2 \hat{\epsilon}_2)\, e^{i\vec{k}\cdot\vec{r} - i\omega t} \tag{14}$$

with complex amplitudes $\mathcal{E}_1$ and $\mathcal{E}_2$

the polarizer corresponding to $\hat{\epsilon}_1$ selects out this field component, i.e., the transmitted field amplitude is

$$E_1 = \hat{\epsilon}_1^* \cdot \vec{E} = \mathcal{E}_1 e^{i\vec{k}\cdot\vec{r} - i\omega t} \tag{15}$$

and so the angular distribution of power measured in *polarization state* $\hat{\epsilon}_1$ is

$$\left(\frac{dP}{d\Omega}\right)_{\mathrm{pol,1}} = \frac{c}{4\pi} |E_1|^2 = \frac{c}{4\pi} |\hat{\epsilon}_1^* \cdot \vec{E}|^2 \tag{16}$$

for *scattering of initially unpolarized* radiation: take average over possible initial polarizations

$$\left(\frac{dP}{d\Omega}\right)_{\mathrm{unpol}} = \frac{1}{2}\left[\left(\frac{dP}{d\Omega}\right)_{\mathrm{pol,init1}} + \left(\frac{dP}{d\Omega}\right)_{\mathrm{pol,init2}}\right] \tag{17}$$

Thomson Scattering

Consider *monochromatic* radiation
linearly polarized in direction $\hat{\epsilon}_{\rm init}$
incident on a free, non-relativistic electron

because non-relativistic, we may ignore magnetic forces *Q: why?*

Q: equation of motion?

Q: and so?

Q: radiation pattern?

magnetic/electric force ratio $F_B/F_E \sim (v/c)B/E = v/c \ll 1$ and so we can ignore F_B

thus the force on the electron is

$$\vec{F} \approx -eE_0\hat{\epsilon}_{\rm init}\ \cos\omega_0 t \tag{18}$$

and thus the electron has

$$\ddot{\vec{r}} = -\frac{e}{m_e}E_0\hat{\epsilon}_{\rm init}\ \cos\omega_0 t \tag{19}$$

and so the dipole moment $\vec{d} = -e\vec{r}$ has

$$\ddot{\vec{d}} = \frac{e^2}{m_e}E_0\hat{\epsilon}_{\rm init}\ \cos\omega_0 t \tag{20}$$

we can solve for the dipole moment

$$\vec{d} = -\frac{e^2 E_0}{m_e\omega_0^2}\hat{\epsilon}_{\rm init}\ \cos\omega_0 t \tag{21}$$

and thus the time-averaged power is

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{e^4 E_0^2}{8\pi m_e^2 c^3} \sin^2 \Theta \tag{22}$$

$$\langle P \rangle = \frac{e^4 E_0^2}{3 m_e^2 c^3} \tag{23}$$

were Θ is angle between $\hat{n}$ and $\hat{a} = \hat{\epsilon}_{\text{init}}$

Q: what's notable about these expressions?

Q: how could we disentangle intrinsic electron response?

Thomson Cross Section

time-averaged power

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{e^4 E_0^2}{8\pi m_e^2 c^3} \sin^2\Theta = \frac{e^4}{m_e^2 c^4} \sin^2\Theta \ \langle S \rangle \qquad (24)$$

where time-averaged incident flux is $\langle S \rangle = cE_0^2/8\pi$

recall: **differential scattering cross section** can be defined as

$$\frac{d\sigma}{d\Omega} = \frac{\text{scattered power}}{\text{incident flux}} = \frac{dP/d\Omega}{\langle S \rangle} \qquad (25)$$

$$= \frac{e^4}{m_e^2 c^4} \sin^2\Theta \qquad (26)$$

integral **Thomson cross section** is

$$\sigma_T \equiv \int \frac{d\sigma}{d\Omega} = \frac{8\pi}{3} \frac{e^4}{m_e^2 c^4} = \frac{8\pi}{3} r_0^2 = 0.665 \times 10^{-24}\ \text{cm}^2 \qquad (27)$$

with the *classical electron radius* $r_0 \equiv e^2/m_e c^2$

Thomson Appreciation

We have found the cross section for scattering of monochromatic, linearly polarized radiation on free electrons:

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{m_e^2 c^4} \sin^2 \Theta \tag{28}$$

$$\sigma = \sigma_{\mathrm{T}} = \frac{8\pi}{3} \frac{e^4}{m_e^2 c^4} \tag{29}$$

Q: notable features?

Q: dependence (or lack thereof) on incident radiation?

plasmas will generally have ions as well as free electrons
Q: which is more important for Thomson scattering?

Q: under what conditions might our assumptions break down?

The Charms of Thomson

Thomson scattering is

- *independent of radiation frequency*
  implicitly assumes electron recoil negligible

$\rightarrow$ initial spectral *shape vs* ν is *unchanged*!

- example: Solar corona highly ionized, Thomson dominates
  Q: implications: spectrum/color? angular distribution?
  Q: how observe? `www: corona`

- $\sigma \propto 1/m^2$: *electron scattering larger than ions*
  by factor $(m_{\text{ion}}/m_e)^2 \gg 10^6$!

- if electron recoil large, and/or electron relativistic
  assumptions break down, will have to revisit

if we measure polarization state $\hat{\epsilon}$,
Q: what is angular pattern of scattered radiation?

in measured = final polarization state $\hat{\epsilon}_{\mathrm{f}}$, find

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{m_e^2 c^4} \left|\hat{\epsilon}_{\mathrm{f}}^* \cdot \hat{\epsilon}_{\mathrm{init}}\right|^2 \tag{30}$$

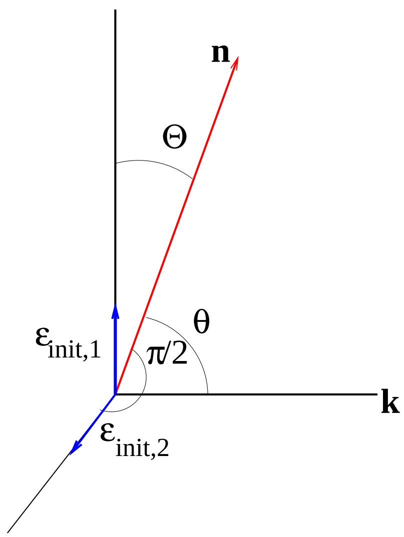
What if radiation is *unpolarized*?

Q: how can we use our result?

Thomson Scattering of Unpolarized Radiation

Using result for linear polarization
we can construct result for unpolarized radiation
by *averaging results for two orthogonal linear polarizations*

Geometry:
$\hat{n}$ is direction of scattered radiation
$\hat{\epsilon}_{\rm init} = \hat{k}$ direction of incident radiation
initial polarizations are both $\perp \hat{k}$
choose one polarization $\hat{\epsilon}_{\rm init,1}$ in $\hat{n} - \hat{k}$ plane
and the other $\hat{\epsilon}_{\rm init,2}$ orthogonal
to this plane and to $\hat{n}$



thus scatter initial polarization 1 by angle $\Theta = \pi/2 - \theta$
and an initial polarization 2 by angle $\pi/2$

thus scatter polarization 1 by angle $\Theta = \pi/2 - \theta$
and polarization 2 by angle $\pi/2$, and so

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} = = \frac{1}{2}\left(\frac{d\sigma}{d\Omega}\right)_1 + \frac{1}{2}\left(\frac{d\sigma}{d\Omega}\right)_2 \tag{31}$$

$$= \frac{r_0^2}{2}\left(1 + \sin^2\Theta\right) \tag{32}$$

$$= \frac{r_0^2}{2}\left(1 + \cos^2\theta\right) \tag{33}$$

which only depends on angle θ
between incident $\widehat{k}$ and scattered $\widehat{n}$ radiation direction

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} = \frac{r_0^2}{2}\left(1 + \cos^2\theta\right) \tag{34}$$

- forward-backward asymmetry: $\theta \to -\theta$ invariance

- angular pattern: $\cos^2\theta \propto \cos 2\theta$ term

$\rightarrow$ scattered radiation has has 180^0 periodicity

$\rightarrow$ a "pole" every 90^0: **quadrupole**

- total cross section $\sigma_{\text{unpol}} = \sigma_{\text{pol}} = \sigma_{\text{T}}$

$\rightarrow$ electron at rest has no preferred direction

- Polarization of scattered radiation

$$\Pi = \frac{1 - \cos^2\theta}{1 + \cos^2\theta} \tag{35}$$

Q: what does this mean?

Thomson Scattering Creates Polarization

Thomson scattering of *initially unpolarized* radiation has

$$\Pi = \frac{1 - \cos^2\theta}{1 + \cos^2\theta} \tag{36}$$

i.e., degree of polarization $P \neq 0$!

Thomson-scattered radiation is linearly polarized!

Quadrupole pattern in angle θ between $\hat{k}_{\rm init}$ and $\hat{n}_{\rm scattered}$

- 100% polarized at $\theta = \pi/2$
- 0% polarized at $\theta = 0, \pi$

classical picture: e^- as dipole antenna

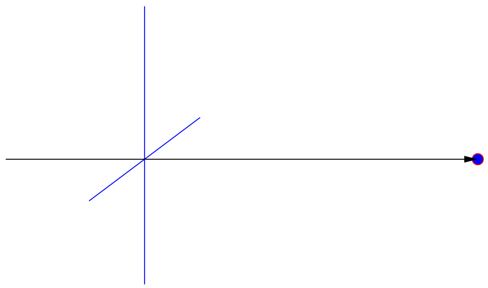
incident linearly polarized wave accelerates e^-

→ $\sin^2\Theta$ pattern, peaks at $\Theta = 0$, i.e., $\|\hat{\epsilon}_{\rm init}$

Thompson Scattering: A Gut Feeling

Discussion swiped from Wayne Hu's website

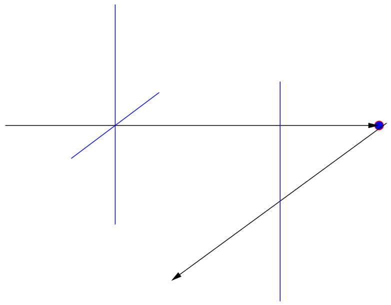
Consider a beam of unpolarized radiation
propagating in plane of sky, incident on an electron
think of as superposition of linear polarizations
one along sightline, one in sky



Q: why is scattered radiation polarized?

Q: now what if unpolarized beams from opposite directions?

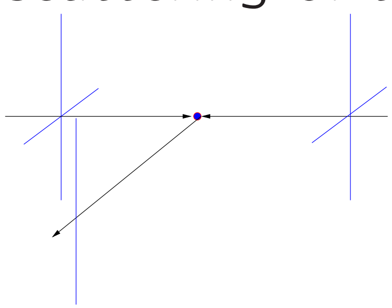
scattering of one unpolarized beam:



$\rightarrow$ see radiation from e motion in sky plane

$\rightarrow$ linear polarization!

scattering of two unpolarized beams in opposite directions:

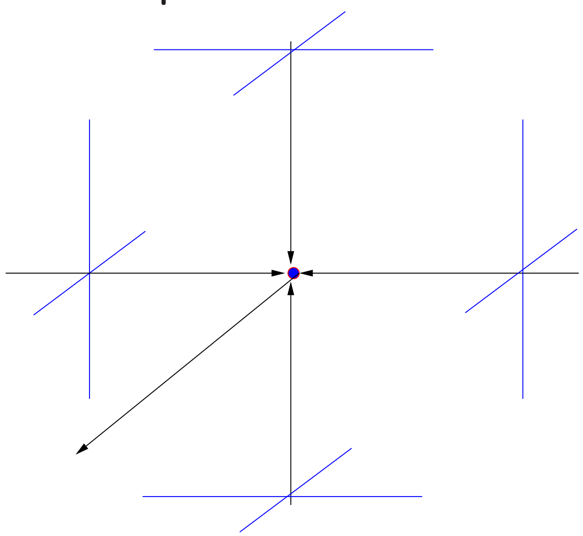


$\rightarrow$ the other side only adds to e motion in sky plane

$\rightarrow$ also linear polarization!

Q: what if isotropic initial radiation field?

isotropic initial radiation field:

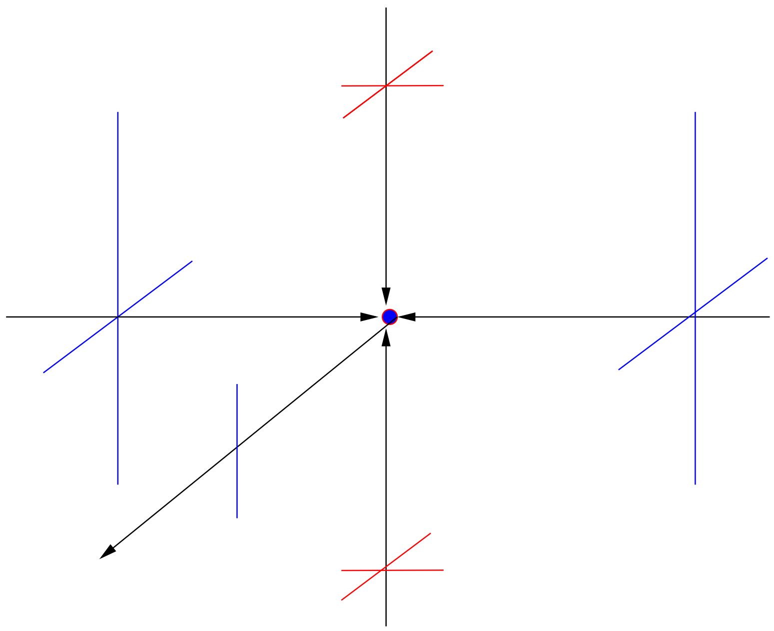


e motions in x and y sky directions cancel
$\rightarrow$ no net polarization

Q: what initial radiation has quadrupole pattern?
i.e., less intense along one axis?

Q: lesson?

if initial radiation field has quadrupole intensity pattern



linear polarization!

lesson: polarization arises from Thomson scattering
when electrons "see" quadrupole anisotropies in radiation field

Awesomest Example of Thompson Polarization: the CMB

The CMB is nearly isotropic radiation field
arises from $\tau = 1$ "surface of last scattering" at $z = 1000$
when free e and protons "re" combined $ep \to H$

- *before recombination:*
  Thomson scattering of CMB photons, Universe opaque
- after recombination: no free e, Universe transparent

consider electron during last scatterings
sees and anisotropic thermal radiation field

consider point at hot/cold "wall"
locally sees *dipole* T anisotropy
net polarization towards us: zero! *Q: why?*

Q: what about edge of circular hot spot? cold spot?

polarization tangential (ring) around hot spots
radial (spokes) around cold spots
(superpose to “+” = zero net polarization–check!)

`www: WMAP polarization observations of hot and cold spots`

Note: polarization & T anisotropies *linked*
$\rightarrow$ consistency test for CMB theory and hence hot big bang

Polarization Observed

First detection: pre-WMAP!
⋆ DASI (2002) ground-based interferometer
at level predicted based on T anisotropies! Woo hoo!

WMAP (2003): first polarization-T correlation function

Planck (March 2013): much more sensitive to polarization
maybe a signature of inflation-generated gravitational radiation?