# **Astronomy 501: Radiative Processes**

Lecture 18

Oct 3, 2022

Announcements:

- Good news: no problem set this week!
- Bad news: **Midterm Exam this Friday**
  Info on Canvas

Last time: finished Thomson scattering

Today: Bremsstrahlung

# Bremsstrahlung

# Bremsstrahlung

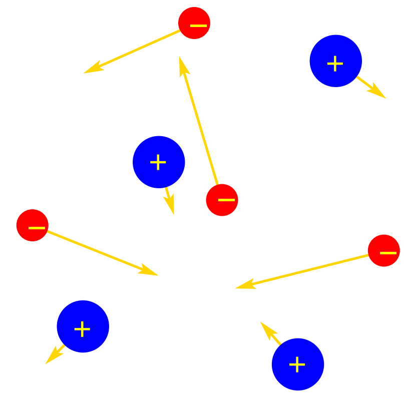
German lesson (Deutschunterricht) for today:

*Bremse* = brake (as in stopping)

*Strahlung* = radiation

and so: Bremsstrahlung = "breaking radiation"

= emission from *decelerated charged particles*



Consider a **dilute plasma** at temperature $T$, with

- free ions: charge $+Ze$, number density $n_{\mathrm{i}}$
- free electrons: charge $-e$, number density $n_e$

*Q: what conditions needed to realize this?*

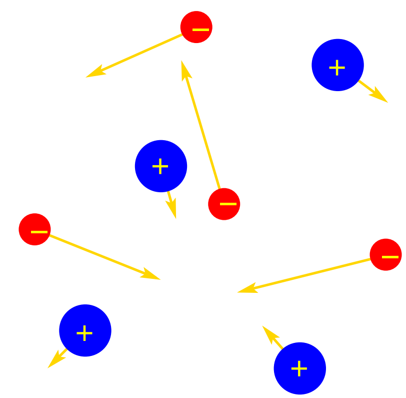
*Q: astrophysical examples?*

*Q: what microphysics what will cause the plasma to emit?*
*i.e., what interactions will occur?*

*Q: which particles will radiate most?*

dilute plasma = low particle density = typical in astrophysics
$\rightarrow$ three-body collisions unlikely; ignore these
$\rightarrow$ focus on **two-body collisions**

possible interactions:
*Coulomb forces* between particle pairs



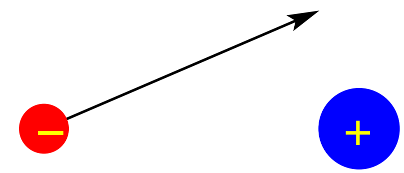
- electron-electron
- ion-ion
- electron-ion

But electrons repel each other!
don't approach closely: *electron-electron acceleration weak*

electron and ion attracted and scattered by same Coulomb force
But $a_i/a_e = m_e/m_i < 10^{-3}$ $\rightarrow$ *ion acceleration negligible*
$\rightarrow$ focus electron acceleration in static field of ion
*electron-ion* radiation dominates

# Coulomb Motion Of An Electron Due To An Ion

consider one *unbound (free) electron*
moving at speed $v_{\text{init}}$ past one ion



*Q: ion's effect on $e^-$ trajectory?*

*Q: $e^-$ acceleration over time?*

*Q: when will radiation be most intense?*

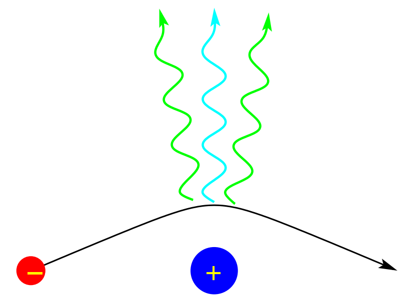
*Q: what $e^-$ trajectories will give the most radiation?*

*Q: compare/contrast with Thomson scattering?*

# Bremsstrahlung From an Electron-Ion Encounter

free $e^-$ feels Coulomb attraction of ion
trajectory deflected: velocity change

acceleration present for all trajectory
but *maximum at closest approach* $r_{\rm min} \equiv b$



$P \propto a^2$ most intense at closest approach

most radiation for max deflection
$\Rightarrow$ smallest $b$, smallest $v_{\rm init}$

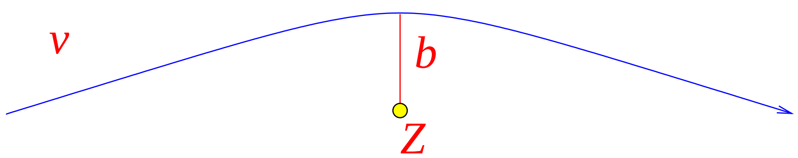
Contrast with Thomson

- bremsstrahlung is *true emission!* **radiation created**!
  **does not** conserve photon number
- acceleration due to Coulomb is not harmonic
  radiation **not** monochromatic! expect **continuous spectrum**

# Order of Magnitude Expectations

start with *classical, nonrelativistic* picture

consider a free, unbound electron with asymptotic speed $v$ moving in Coulomb field of stationary ion




let $b =$ *the distance of closest approach* or **impact parameter**

*Q: estimate of maximum acceleration?*

*Q: duration of acceleration? velocity change? radiation frequency*

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Recall the *Spirit of Order-of-Magnitude*:

- ignore all dimensionless constants, e.g., "small circle approximation" $2\pi \approx 1$
- lower your expectations for precision
- use rough result to guide more careful calculations

maximum acceleration:

Coulomb acceleration at closest approach

$$a_{\text{max}} \sim \frac{Ze^2}{m_e b^2} \tag{1}$$

duration of acceleration: **collision time**

$$\tau \sim \frac{b}{v} \tag{2}$$

velocity change

$$\Delta v \sim a_{\text{max}}\,\tau \sim \frac{Ze^2}{m_e v b} \sim \left(\frac{Ze^2/b}{m_e v^2}\right) v \tag{3}$$

frequency of radiation: use only timescale in problem

$$\omega \sim \frac{1}{\tau} \sim \frac{v}{b} \tag{4}$$

∞ *Q: what is maximum radiated power? radiated energy? energy per unit freq?*

estimate maximum radiated power to be

$$P_{\text{max}} \sim \frac{e^2 a_{\text{max}}^2}{c^3} \sim \frac{e^2 \Delta v^2}{c^3 \ \tau^2} \sim \frac{Z^2 e^6}{m_e^2 v^2 b^2 \ \tau^2} \tag{5}$$

radiated energy

$$\Delta W \sim P_{\text{max}} \ \tau \sim \frac{Z^2 e^6}{m_e^2 v^2 b^2 \ \tau} \tag{6}$$

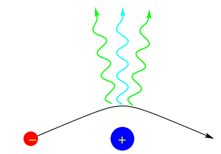
radiated energy per unit frequency

$$\frac{\Delta W}{\Delta \nu} \sim \frac{\Delta W}{\omega} \sim \frac{Z^2 e^6}{m_e^2 v^2 b^2} \tag{7}$$

- this is energy radiated in **one** electron-ion encounter having closest approach $b$
- note that $\Delta W \propto 1/vb^3$: largest emission for small $v$ and $b$, as anticipated
- note $dW/d\omega$ indep of $\omega$: broad spectrum, as anticipated

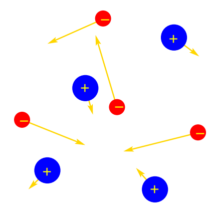
# Multiple Encounters

so far: radiation from a single $e^-$-ion interaction
where closest approach was $b$



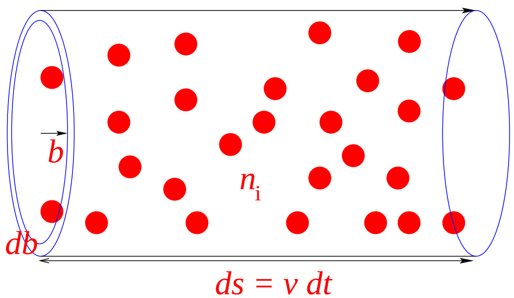
$\Downarrow$

now: sum interactions with many ions
having many different $b$



electron with speed $v$ moves encounters ion number density $n_\mathrm{i}$

- we want number of ions $d\mathcal{N}_\mathrm{i}$ that $e$ encounters
  out to distance $\sim b$ in time $dt$ *Q: which is?*
  hint–think of $e^-$ sweeping area $\pi b^2$ as it moves
- *Q: what is typical rate of energy emitted per electron?*




in cylindrical distance $(b, b + db)$, *volume swept* is

$$dV = 2\pi\ b\ db\ ds = 2\pi\ v\ b\ db\ dt \tag{8}$$

i.e., $dV \sim b^2\ v\ dt$

thus number of ions encountered is

$$d\mathcal{N}_{\rm i} = n_{\rm i}\ dV\ \ \sim n_{\rm i}\ b^2\ v\ dt \tag{9}$$

Thus the rate of energy emitted = *power emitted per* $e$ is

$$\frac{dP_{{\rm per}e}}{d\nu} = \frac{\Delta W}{\Delta \nu}\ \frac{d\mathcal{N}_{\rm i}}{dt} \sim \frac{e^6 Z^2}{m_e c^3 v} n_{\rm i} \tag{10}$$

*Q: and so what is emission coefficient* $j_\nu$*?*

Our order-of-magnitude estimate for the emission coefficient from nonrelativistic bremsstrahlung:

$$j_\nu = n_e \frac{dP_{\mathrm{per}\,e}}{d\nu} \sim \frac{e^6 Z^2}{m_e c^3 v} n_e n_{\mathrm{i}} \quad (11)$$

*Q: now what's the basic physical picture?*

*Q: notable features? what didn't we get from order of mag?*

*Q: how can we do the classical calculation more carefully?*

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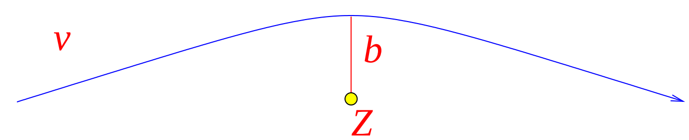
but first, for refreshement, recall:

*Q: what astrophysical situations give rise to bremsstrahlung?*

`www: awesome examples`

# Bremsstrahlung = Free-Free Emission: Physical Picture

motion of *free* electrons
through a plasma of (*free*) ions




we approximate this as a series of

- *two-body electron-ion* scattering events
- *unbound Coulomb* trajectories: *hyperbolæ*
  $\rightarrow$ asymptotically free, scattered through small angle
- acceleration maximum at closest approach $b$
  lasting for scattering time $\tau = b/v$
- burst of radiation over this time, frequency $\nu \sim 1/\tau$

So net effect is

- many scattering events
- a series of small-angle scatterings
- and radiation bursts at different frequencies

Our order-of-magnitude estimate for the emission coefficient from nonrelativistic bremsstrahlung:

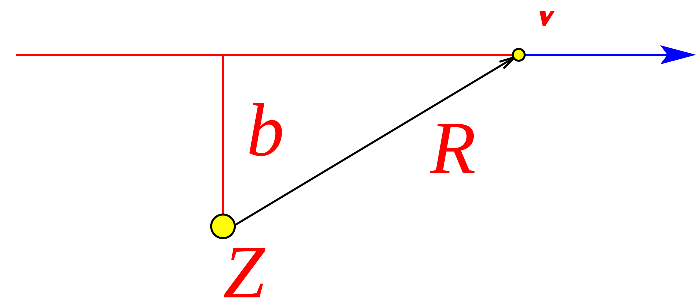
$$j_\nu = n_e \frac{dP_{\rm per}e}{d\nu} \sim \frac{e^6 Z^2}{m_e c^3 v} n_e n_{\rm i} \tag{12}$$

*Q: notable features? what didn't we get from order of mag?*

*Q: how can we do the classical calculation more carefully?*

# Bremsstrahlung: Classical Calculation

Consider electron with initial speed $v$
with *impact parameter* $b$
moving fast enough so that
*scattered through small angle*




dipole moment $\vec{d} = -e\vec{R}$, with second derivative

$$\ddot{\vec{d}} = -e\dot{\vec{v}} \tag{13}$$

take Fourier transform

$$-\omega^2 \, \vec{\tilde{d}}(\omega) = -\frac{e}{2\pi} \int_{-\infty}^{\infty} \dot{\vec{v}}(t) \; e^{i\omega t} \; dt \tag{14}$$

where: $\vec{v}(t)$ is *an unbound Coulomb trajectory*:
$\rightarrow$ *hyperbola* in space, complicated function of time
but: $\dot{\vec{v}}(\omega)$ simplifies in limiting cases
$\rightarrow$ compare $\omega$ and collision time $\tau = b/v$
*Q:* $\omega\tau \gg 1$? $\omega\tau \ll 1$?

$$-\omega^2\ \vec{\tilde{d}} = -\frac{e}{2\pi}\int_{-\infty}^{\infty} \dot{\vec{v}} e^{i\omega t}\ dt \tag{15}$$

but $\vec{v}(t)$ only changes on timescale $\tau$:
for $\omega\tau \gg 1$, many oscillations during acceleration
complex phase averages out: $\vec{v}(\omega) \to 0$

for $\omega\tau \ll 1$, complex exponent unchanged during accel
phase unimportant: $\vec{v}(\omega) \to \int \dot{\vec{v}}\ dt = \Delta\vec{v}$

and thus only frequencies $\omega \lesssim \tau^{-1} = b/v$ contribute
and the dipole moment has

$$\vec{d}(\omega) \to \begin{cases} \frac{e}{2\pi\omega^2}\Delta\vec{v} & \omega\tau \ll 1 \\ 0 & \omega\tau \gg 1 \end{cases} \tag{16}$$

Energy emitted per unit frequency at impact parameter $b$:

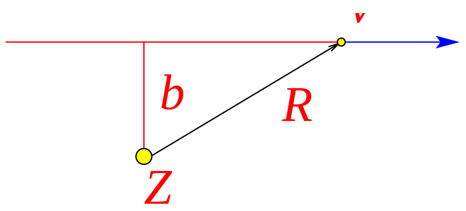
$$\frac{dW}{d\omega} = \frac{8\pi e^2 \omega^4 |d(\omega)|^2}{3c^3} \rightarrow \begin{cases} \frac{2e^2}{3\pi c^3}|\Delta \vec{v}|^2 & \omega\tau \ll 1 \\ 0 & \omega\tau \gg 1 \end{cases} \tag{17}$$

Now find $\Delta\vec{v}$: for small deflection

$$\Delta v \approx \Delta v_\perp = \int F_z \, dt \tag{18}$$

$$= \frac{Ze^2}{m_e}\int \frac{b}{(b^2 + v^2t^2)^{3/2}} dt \tag{19}$$

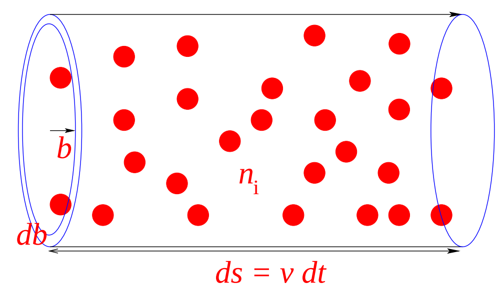
$$= \frac{2Ze^2}{m_e b v} \tag{20}$$



Dipole formula give energy emitted per electron scattering at $b$

$$\frac{dW(b)}{d\omega} = \begin{cases} \frac{8Z^2e^2}{3\pi c^3 m_e^2 v b^2} & \omega\tau \ll 1 \\ 0 & \omega\tau \gg 1 \end{cases} \tag{21}$$

to include all impact parameters $b$:
weight by *collision rate*
$d\mathcal{N}_i/dt = 2\pi n_i v \; b\, db$ per electron
gives power emitted power per volume




$$\frac{dW(b)}{dV\; d\omega\; dt} = n_e \frac{dW}{d\omega}\frac{d\mathcal{N}_{\rm i}}{dt} = 2\pi n_e n_{\rm i} \int_{b_{\rm min}}^{b_{\rm max}} \frac{dW(b)}{d\omega}\; b\; db \tag{22}$$

*Q: what will be $b$ dependence?*

using low-frequency result:

$$q_\nu = 4\pi j_\nu = \frac{dW}{dV\ d\omega\ dt} = \frac{16 Z^2 e^6}{3\pi c^3 m_e^2 v} n_e n_{\rm i}\ \ln\left(\frac{b_{\rm max}}{b_{\rm min}}\right) \tag{23}$$

compare/contrast with order-of-magnitude:

- $j_\nu \propto n_e n_i$: linear scaling with $e$ and ion density
- $j_\nu \propto 1/v$ scaling
- independence of $b$ range $\rightarrow$ log dependence
- independence with $\nu, \omega$: "flat" emission spectrum

# Impact Parameter Range

bremsstrahlung emission at speed $v$, frequency $\omega$
depends *logarithmically* on the limits
$b_{\rm min}, b_{\rm max}$ of impact parameter
within our classical, small-angle-scattering treatment

*lower limit*

- quantum mechanics: $\Delta x \ \Delta p \gtrsim \hbar$
  $\rightarrow b_{\rm min}^{(1)} > h/mv$
- small-angle: $\Delta v/v \sim Ze^2/bmv^2 < 1$
  $\rightarrow b_{\rm min}^{(2)} > Ze^2/mv^2$

*upper limit*

for a fixed $\omega$ and $v$, max impact parameter is $b_{\rm max} \sim v/\omega$

fortunately: log dependence on limits
$\rightarrow$ results not very sensitive to details of choices

# Single-Velocity Bremsstrahlung

convenient, conventional form for bremsstrahlung emission also known as **free-free** emission

$$4\pi \; j_\omega(\omega, v) = \frac{16\pi}{3\sqrt{3}} \frac{Z^2 e^6}{m_e^2 c^3 v} \; n_\mathrm{i} n_e \; g_\mathrm{ff}(\omega, v) \tag{24}$$

eeq

uses the dimensionless correction factor or **Gaunt factor**

$$g_\mathrm{ff}(\omega, v) = \frac{\sqrt{3}}{\pi} \; \ln\left(\frac{b_\mathrm{max}}{b_\mathrm{min}}\right) \tag{25}$$

- accounts for log factor
- typically $g_\mathrm{ff} \sim 1$ to few
- tables and plots available



`www: awesome astrophysical example`

*Q: how does this differ from our treatment?*