Astronomy 501: Radiative Processes

Lecture 19

Oct 5, 2022

Announcements:

- **Midterm Exam this Friday**, Info on Canvas
  will discuss at end of class today
- Office Hours: after class or by appointment

Last time: Bremsstrahlung

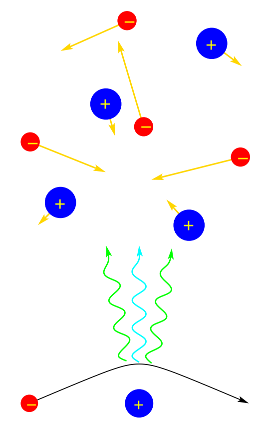
Q: what are actors–which particles? what interaction?

Q: key timescale?

Q: contrast with Thomson?

Bremsstrahlung = Free-Free Emission: Physical Picture

motion of *free* electrons
through a plasma of (*free*) ions



we approximate this as a series of

- *two-body electron-ion* scattering events
- *unbound Coulomb* trajectories: *hyperbolæ*

  $\rightarrow$ asymptotically free, scattered through small angle
- acceleration maximum at closest approach b
  lasting for scattering time $\tau = b/v$
- burst of radiation over this time, frequency $\nu \sim 1/\tau$

So net effect is

- many scattering events
- a series of small-angle scatterings
- and radiation bursts at different frequencies

Our order-of-magnitude estimate for the emission coefficient from nonrelativistic bremsstrahlung:

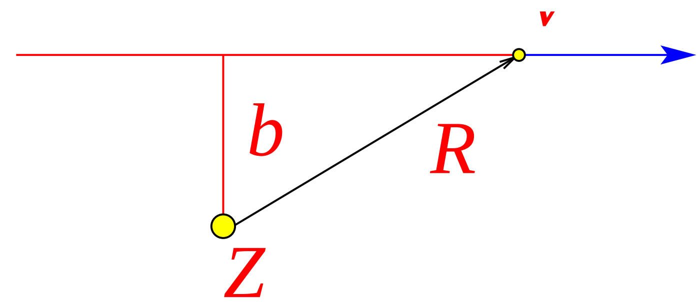
$$j_\nu = n_e \frac{dP_{\text{per}\,e}}{d\nu} \sim \frac{e^6 Z^2}{m_e c^3 v} n_e n_{\text{i}} \tag{1}$$

Contrast with Thomson

- bremsstrahlung is *true emission!* **radiation created***!*
  does not conserve photon number
- acceleration due to Coulomb is not harmonic
  radiation **not** monochromatic! expect **continuous spectrum**

Bremsstrahlung: Classical Calculation

Consider electron with initial speed v
with *impact parameter* b
moving fast enough so that
scattered through small angle



dipole moment $\vec{d} = -e\vec{R}$, with second derivative

$$\ddot{\vec{d}} = -e\dot{\vec{v}} \tag{2}$$

take Fourier transform

$$-\omega^2 \ \tilde{\vec{d}} = -\frac{e}{2\pi}\int_{-\infty}^{\infty} \dot{\vec{v}}(t) \ e^{i\omega t} \ dt \tag{3}$$

where: $\vec{v}(t)$ is *an unbound Coulomb trajectory*:
$\rightarrow$ *hyperbola* in space, complicated function of time
but: $\dot{\vec{v}}(\omega)$ simplifies in limiting cases
$\rightarrow$ compare ω and collision time $\tau = b/v$
Q: $\omega\tau \gg 1$? $\omega\tau \ll 1$?

$$-\omega^2\ \vec{d}(\omega) = -\frac{e}{2\pi}\int_{-\infty}^{\infty} \dot{\vec{v}} e^{i\omega t}\ dt \tag{4}$$

but $\vec{v}(t)$ only changes on timescale τ:
for $\omega\tau \gg 1$, many oscillations during acceleration
complex phase averages out: $\vec{v}(\omega) \to 0$

for $\omega\tau \ll 1$, complex exponent unchanged during accel
phase unimportant: $\vec{v}(\omega) \to \int \dot{\vec{v}}\ dt = \Delta\vec{v}$

and thus only frequencies $\omega \lesssim \tau^{-1} = b/v$ contribute
and the dipole moment has

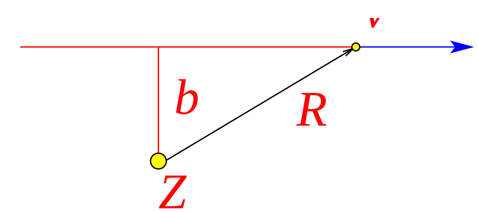
$$\vec{d}(\omega) \to \begin{cases} \frac{e}{2\pi\omega^2}\Delta\vec{v} & \omega\tau \ll 1 \\ 0 & \omega\tau \gg 1 \end{cases} \tag{5}$$

Energy emitted per unit frequency at impact parameter b:

$$\frac{dW}{d\omega} = \frac{8\pi e^2 \omega^4 \, |d(\omega)|^2}{3c^3} \rightarrow \begin{cases} \frac{2e^2}{3\pi c^3} |\Delta \vec{v}|^2 & \omega\tau \ll 1 \\ 0 & \omega\tau \gg 1 \end{cases} \tag{6}$$

Now find $\Delta \vec{v}$: for small deflection

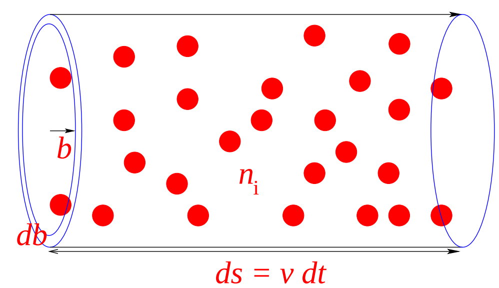
$$\begin{aligned} \Delta v &\approx \Delta v_\perp = \int F_z \, dt & (7) \\ &= \frac{Ze^2}{m_e} \int \frac{b}{(b^2 + v^2 t^2)^{3/2}} dt & (8) \\ &= \frac{2Ze^2}{m_e b v} & (9) \end{aligned}$$



Dipole formula give energy emitted per electron scattering at b

$$\frac{dW(b)}{d\omega} = \begin{cases} \frac{8Z^2e^2}{3\pi c^3 m_e^2 v b^2} & \omega\tau \ll 1 \\ 0 & \omega\tau \gg 1 \end{cases} \tag{10}$$

to include all impact parameters b:
weight by *collision rate*
$d\mathcal{N}_i/dt = 2\pi n_i v \; b \, db$ per electron
gives power emitted power per volume



$$\frac{dW(b)}{dV \; d\omega \; dt} = n_e \frac{dW}{d\omega}\frac{d\mathcal{N}_\mathrm{i}}{dt} = 2\pi n_e n_\mathrm{i} \int_{b_\mathrm{min}}^{b_\mathrm{max}} \frac{dW(b)}{d\omega} \; b \; db \tag{11}$$

Q: what will be b dependence?

using low-frequency result:

$$q_\nu = 4\pi j_\nu = \frac{dW}{dV\ d\omega\ dt} = \frac{16 Z^2 e^6}{3\pi c^3 m_e^2 v} n_e n_{\rm i}\ \ln\left(\frac{b_{\rm max}}{b_{\rm min}}\right) \tag{12}$$

compare/contrast with order-of-magnitude:

- $j_\nu \propto n_e n_i$: linear scaling with e and ion density
- $j_\nu \propto 1/v$ scaling
- independence of b range $\rightarrow$ log dependence
- independence with ν, ω: “flat” emission spectrum

Impact Parameter Range

bremsstrahlung emission at speed v, frequency ω
depends *logarithmically* on the limits
$b_{\rm min}, b_{\rm max}$ of impact parameter
within our classical, small-angle-scattering treatment

lower limit

- quantum mechanics: $\Delta x \ \Delta p \gtrsim \hbar$
  $\rightarrow b_{\rm min}^{(1)} > h/mv$
- small-angle: $\Delta v/v \sim Ze^2/bmv^2 < 1$
  $\rightarrow b_{\rm min}^{(2)} > Ze^2/mv^2$

upper limit

for a fixed ω and v, max impact parameter is $b_{\rm max} \sim v/\omega$

fortunately: log dependence on limits
$\rightarrow$ results not very sensitive to details of choices

Single-Velocity Bremsstrahlung

convenient, conventional form for bremsstrahlung emission also known as **free-free** emission

$$4\pi\ j_\omega(\omega, v) = \frac{16\pi}{3\sqrt{3}} \frac{Z^2 e^6}{m_e^2 c^3 v}\ n_{\mathrm{i}} n_e\ g_{\mathrm{ff}}(\omega, v) \tag{13}$$

eeq

uses the dimensionless correction factor or **Gaunt factor**

$$g_{\mathrm{ff}}(\omega, v) = \frac{\sqrt{3}}{\pi}\ \ln\left(\frac{b_{\mathrm{max}}}{b_{\mathrm{min}}}\right) \tag{14}$$

- accounts for log factor
- typically $g_{\mathrm{ff}} \sim 1$ to few
- tables and plots available

`www: awesome astrophysical example`

Q: how does this differ from our treatment?

Thermal Bremsstrahlung

so far: calculated bremsstrahlung emission for
a *single electron velocity* v
$\rightarrow$ a "beam" of mono-energetic electrons

but in real astrophysical applications
there is a *distribution* of electron velocities
usually: a *thermal* distribution

so we wish to find
the *mean* or *expected* emission $\langle j_{\nu,\text{brem}} \rangle$
for a thermal distribution of velocities

Q: order-of-magnitude expectation?

Thermal Bremsstrahlung: Order-of-Magnitude

order-of-magnitude emission for single v:

$$j_\nu \sim \frac{e^6 Z^2}{m_e c^3 v} n_e n_{\rm i} \tag{15}$$

i.e., $j_\nu \sim 1/v$

thus, thermal average

$$\langle j_\nu \rangle \sim \frac{Z^2 e^6}{m_e c^3 v_T} n_e n_{\rm i} \tag{16}$$

with v_T a typical thermal velocity

find v_T from equipartition: $m_e v_T^2 \sim kT \to v_T \sim \sqrt{kT/m_e}$

Q: how do we approach the honest, detailed calculation?

Q: yet more new formalism?

Thermal Particles: Non-Relativistic Limit

recall: semiclassically, particle behavior in *phase space* $(\vec{x}, \vec{p})$
described by *distribution function* f:

- Heisenberg: minimum phase-space "cell" size $dx\ dp = h$
- particle number $dN = g/h^3\ f(\vec{x}, \vec{p})\ d^3\vec{x}\ d^3\vec{p}$

a *dilute*=non-degenerate, *non-relativistic* particle species
of mass m at temperature T
has distribution function

$$f_{\rm therm}(p) \propto e^{-p^2/2mT} \tag{17}$$

and thus has number density $n \propto \int e^{-p^2/2m_eT} d^3\vec{p} \propto \int e^{-m_e v^2/2kT} d^3\vec{v}$

Q: how to compute thermal averaged bremsstrahlung emission?

Bremsstrahlung emissivity depends on electron properties via

$$j_\nu(\nu, T) = \langle j_\nu(\nu, v) \rangle \propto \left\langle \frac{g_{\mathrm{ff}}(\nu, v)\ n_e}{v} \right\rangle \tag{18}$$

where

$$\left\langle \frac{g_{\mathrm{ff}}(\omega, v)\ n_e}{v} \right\rangle \sim \int_{v_{\mathrm{min}}}^{\infty} \frac{g_{\mathrm{ff}}(\omega, v)}{v}\ e^{-m_e v^2/2kT}\ d^3\vec{v} \tag{19}$$

Note lower limit v_{min} at fixed ν
$\rightarrow$ minimum electron velocity needed to radiate photon of energy ν

Q: what value should this have? effect on final result?

energy conservation: to make photon of frequency ν electron needs kinetic energy $m_e v^2/2 > h\nu$, so

$$v_{\rm min} = \sqrt{\frac{2h\nu}{m_e}} \tag{20}$$

thus exponential factor has

$$e^{-\frac{m_e v^2}{2kT}} = e^{-\frac{m_e v_{\rm min}^2}{2kT}} \; e^{-\frac{m_e(v^2-v_{\rm min}^2)}{2kT}} = e^{-\frac{h\nu}{kT}} \; e^{-\frac{m_e(v^2-v_{\rm min}^2)}{2kT}}$$

$\rightarrow$ overall factor $e^{-h\nu/kT}$ in thermal average
$\rightarrow$ photon production thermally suppressed at $h\nu > kT$

thermal bremsstrahlung = "free-free" emission result:

$$4\pi j_{\nu,{\rm ff}}(T) = \frac{2^5\pi \; Z^2 \; e^6}{3 \; m_e c^3} \left(\frac{2\pi}{3m_e kT}\right)^{1/2} \; \bar{g}_{\rm ff}(\nu,T) \; e^{-h\nu/kT} \; n_e \; n_{\rm i} \tag{21}$$

with $\bar{g}_{\rm ff}(\nu,T)$ the *velocity-averaged thermal Gaunt factor*
Q: spectral shape for optically thin plasma? implications?
Q: integrated emission?

$$4\pi j_{\nu,\mathrm{ff}}(T) = \frac{2^5\pi\ Z^2\ e^6}{3\ m_e c^3} \left(\frac{2\pi}{3 m_e kT}\right)^{1/2}\ \bar{g}_{\mathrm{ff}}(\nu, T)\ e^{-h\nu/kT}\ n_e\ n_{\mathrm{i}} \quad (22)$$

main frequency dependence is $j_\nu \propto e^{-h\nu/kT}$

$\rightarrow$ **flat spectrum**, cut off at $\nu \sim kT/h$

$\rightarrow$ can use to determine temperature of hot plasma (PS6)

integrated bremsstrahlung emission:

$$4\pi j_{\mathrm{ff}}(T) = 4\pi \int j_{\nu,\mathrm{ff}}(T)\ d\nu \quad (23)$$

$$= \frac{2^5\pi\ Z^2\ e^6}{3\ h m_e c^3} \left(\frac{2\pi kT}{3 m_e}\right)^{1/2}\ \bar{g}_{\mathrm{B}}(T)\ e^{-h\nu/kT}\ n_e\ n_{\mathrm{i}} \quad (24)$$

$$= 1.4 \times 10^{-27}\ \mathrm{erg\ s^{-1}\ cm^{-3}}\ \bar{g}_{\mathrm{B}} \left(\frac{T}{\mathrm{K}}\right)^{\frac{1}{2}} \left(\frac{n_e}{1\ \mathrm{cm^{-3}}}\right) \left(\frac{n_{\mathrm{i}}}{1\ \mathrm{cm^{-3}}}\right)$$

with $\bar{g}_{\mathrm{B}}(T) \sim 1.2 \pm 0.2$ a frequency-averaged Gaunt factor

Q: all of this was for emission–what about the thermal bremsstrahlung absorption coefficient?

Bremsstrahlung Summary

for a *fixed electron velocity* v

$$4\pi \ j_\omega(\omega, v) = \frac{16\pi}{3\sqrt{3}} \frac{Z^2 e^6}{m_e^2 c^3 v} \ n_{\mathsf{i}} n_e \ g_{\mathsf{ff}}(\omega, v) \tag{25}$$

for a **thermal distribution of velocities** = Maxwell-Boltzmann

$$4\pi j_{\nu,\mathsf{ff}}(T) = \frac{2^5 \pi \ Z^2 \ e^6}{3 \ m_e c^3} \left(\frac{2\pi}{3 m_e k T} \right)^{1/2} \ \bar{g}_{\mathsf{ff}}(\nu, T) \ e^{-h\nu/kT} \ n_e \ n_{\mathsf{i}} \tag{26}$$

with *Gaunt factor* $g_{\mathsf{ff}} \sim 1$

Q: spectral features?

Q: all of this was for emission–what about the thermal bremsstrahlung absorption coefficient $\alpha_\nu(T)$*?*

Thermal Bremsstrahlung Absorption

for *thermal* system, *Kirchoff's law:* $S_\nu = B_\nu(T) = j_\nu/\alpha_\nu$

thus we have

$$\alpha_{\nu,\mathrm{ff}} = \frac{j_{\nu,\mathrm{ff}}}{B_\nu(T)} = \frac{4\ Z^2\ e^6}{3\ m_e hc}\left(\frac{2\pi}{3m_e kT}\right)^{1/2} \bar{g}_{\mathrm{ff}}(\nu,T)\ \nu^{-3}\ \left(1 - e^{-h\nu/kT}\right) n_e\, n_{\mathrm{i}}$$

- note that $\alpha_{\nu,\mathrm{ff}} \propto n_e n_{\mathrm{i}}$: two factors of density!
- contrasts with usual $\alpha_\nu = n\sigma_\nu \propto n$: *one* factor of density

Why? *brems. absorption is a 3-body process:* $\gamma + e^- +$ ion
photon absorption depends on both n_e and n_i

Bremsstrahlung Absorption: Frequency Dependence

$$\alpha_{\nu,\mathrm{ff}}(T) = \frac{4\ Z^2\ e^6}{3\ m_e hc}\left(\frac{2\pi}{3m_e kT}\right)^{1/2}\ \bar{g}_{\mathrm{ff}}(\nu,T)\ \nu^{-3}\ \left(1-e^{-h\nu/kT}\right)\ n_e\ n_{\mathrm{i}}$$

limits:

- $h\nu \ll kT$: $\alpha_{\nu,\mathrm{ff}} \propto \nu^{-2}$
- $h\nu \gg kT$: $\alpha_{\nu,\mathrm{ff}} \propto \nu^{-3}$

Q: sketch optical depth vs ν? implications?

Bremsstrahlung Self-Absorption

bremsstrahlung optical depth at small ν:

$$\tau_\nu \propto \alpha_{\nu,\mathrm{ff}} \propto \nu^{-2} \tag{27}$$

guaranteed optically thick below some ν
$\rightarrow$ free-free emission is absorbed inside plasma:
bremsstrahlung self-absorption

thus observed plasma spectra should have three regimes
- small ν: $\tau_\nu \gg 1$, optically thick, $I_\nu \rightarrow B_\nu \propto \nu^3$
- $h\nu < kT$: optically thin, $I_\nu \rightarrow j_\nu s$ *flat* vs ν
- $h\nu \gg kT$: thermally suppressed, $I_\nu \rightarrow j_\nu s \sim e^{-h\nu/kT}$

Q: expected X-ray count spectrum for galaxy cluster?
`www: observations`

Bremsstrahlung and Stellar Interiors

in stellar interiors, bremsstrahlung known as **free-free** emission

stellar flux transfer depends on frequency-averaged opacity

Q: what's opacity?

Q: how do we perform frequency average?

Bremsstrahlung and Stellar Interiors

opacity defined via

$$\alpha_\nu = n\sigma_\nu = \rho\kappa_\nu \tag{28}$$

and in thermal radiation field, frequency avg is **Rosseland mean**

$$\frac{1}{\alpha_{\rm R}} = \frac{\int (\alpha_\nu + \varsigma_\nu)^{-1} \partial_T B_\nu \, d\nu}{\int \partial_T B_\nu \, d\nu} \tag{29}$$

for bremsstrahlung at $h\nu < kT$

$$\alpha_\nu \sim \frac{Z^2 \nu^2}{T^{3/2}} \, n_e \, n_i \tag{30}$$

Q: and so what do we expect for $\alpha_{\rm R}(\rho, T)$*?*

Free-Free Opacity

free-free = bremsstrahlung: $\alpha_\nu \sim \nu^2/T^{3/2}\ n_e\ n_i$
scalings in Rosseland mean for bremsstrahlung:

- $h\nu \sim kT$
- $n_i \propto \rho$, and $n_e \propto \rho/\mu_e$

gives **Kramer's Law** for opacity

$$\kappa_{\rm ff}(\rho, T) \sim \frac{\rho}{\mu_e T^{7/2}} \tag{31}$$

appears in local flux expression

$$F(z) = -\frac{16\sigma T^3}{3\kappa\rho}\frac{\partial T}{\partial z} \tag{32}$$

important for stellar interiors
bonus: same scaling works for bound-free!

Build Your Toolbox: Bremsstrahlung

microphysics: matter-radiation interactions

Q: physical origin of bremsstrahlung?

Q: physical nature of sources?

Q: spectrum characteristics?

Q: frequency range?

real/expected astrophysical sources of Thomson scattering

Q: where do we expect this to be important?

Q: relevant EM bands? temperatures?

Toolbox: Bremsstrahlung

emission physics

- physical origin: non-relativistic Coulomb acceleration in free electrons and ions interactions
- physical sources: need free e^- and ions $\rightarrow$ ionized gas= plasma
- spectrum: continuum emission
  for thermal plasma, $j_\nu \sim const$ vs ν
  up to *exponential cutoff* $e^{-h\nu/kT}$

astrophysical sources of bremsstrahlung emission/absorption

- sites are highly ionized gas: stellar interiors.
  hot nebulæ: accretion disks, Hii regions, supernova remnants, galaxy clusters. the early Universe.
- EM bands radio to X-ray
- temperatures hot enough to be ionized: $T \gtrsim 10^5$ K