Astronomy 501: Radiative Processes

Lecture 20

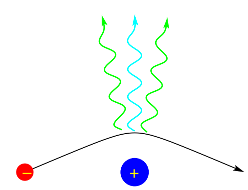
Oct 10, 2022

Announcements:

- **Problem Set 6 due Friday**
- Grading Elf (=BDF) at work on exam

Last time: Bremsstrahlung

- true emission process in ionized gas: plasma
- e^-+ion "flyby" $\Rightarrow$ acceleration $\Rightarrow$ radiation burst
- emission rate $\propto$ encounter rate



Q: spectrum vs ν or $\omega = 2\pi\nu$?

for single encounter of e^- with initial speed v
and *impact parameter* b

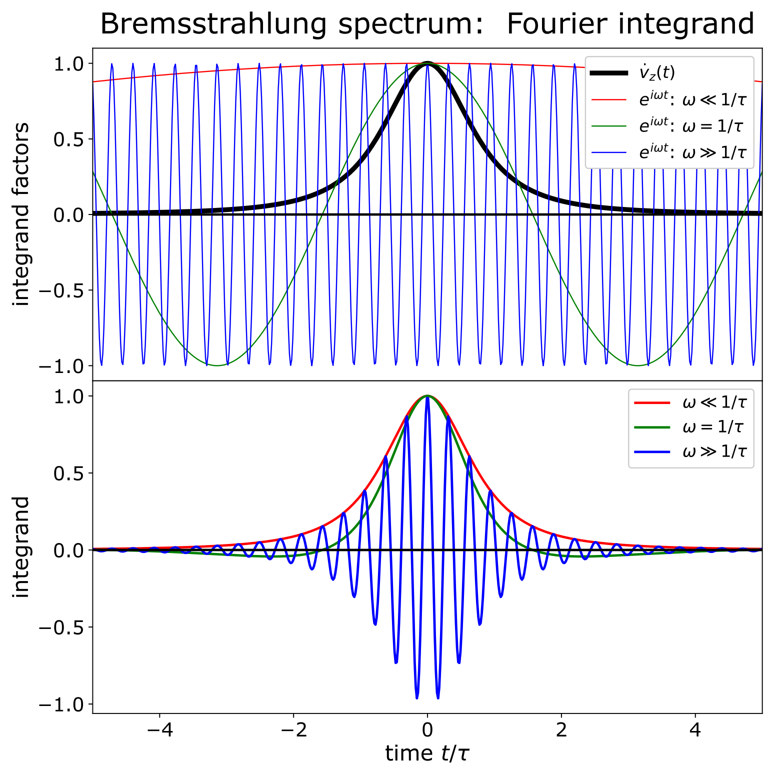
spectrum of emission from electron: $dW/d\omega = 2|\ddot{\tilde{d}}(\omega)|^2/3c^2$
take Fourier transform

$$\ddot{\vec{d}}(\omega) \approx -\frac{e}{2\pi}\int_{-\infty}^{\infty} \dot{\vec{v}}_\perp(t)\ e^{i\omega t}\ dt \qquad (1)$$

can rewrite as

$$\ddot{d}_\perp \approx \frac{Ze^2}{2\pi m_e b v}\int_{-\infty}^{\infty} \frac{e^{i\omega\tau u}}{1+u^2}\ du \qquad (2)$$

where dimensionless integral depends on $\omega\tau = \omega b/v$
different behavior if $\omega \ll 1/\tau$ versus $\omega \gg 1/\tau$



$$\ddot{d}_\perp \approx \frac{Ze^2}{2\pi m_e b v} \int_{-\infty}^{\infty} \frac{e^{i\omega\tau u}}{1+u^2}\, du \tag{3}$$

Lessons:

- for $\omega \lesssim 1/\tau$, integral is large and nearly uniform
- for $\omega \gg 1/\tau$ integral oscillates $\to 0$

spectrum for a single e-ion flyby is:

- **continuous**
- **roughly flat for** $\omega \lesssim 1/\tau$
- **cut off at** $\omega \gtrsim 1/\tau$

ω

Thermal Bremsstrahlung

Now average single-velocity emission over thermal distribution of speeds • to emit photon at ν demands minimum e^- speed

$$v_{\rm min} = \sqrt{\frac{2h\nu}{m_e}} \tag{4}$$

• gives overall factor $e^{-h\nu/kT}$ in thermal average
• photon production thermally suppressed at $h\nu > kT$

thermal bremsstrahlung = "free-free" emission result:

$$4\pi j_{\nu,\rm ff}(T) = \frac{2^5\pi\ Z^2\ e^6}{3\ m_e c^3}\left(\frac{2\pi}{3m_e kT}\right)^{1/2}\ \bar{g}_{\rm ff}(\nu,T)\ e^{-h\nu/kT}\ n_e\ n_{\rm i} \tag{5}$$

with $\bar{g}_{\rm ff}(\nu,T)$ the *velocity-averaged thermal Gaunt factor*

Q: spectral shape for optically thin plasma? implications?

Q: integrated emission?

$$4\pi j_{\nu,\mathrm{ff}}(T) = \frac{2^5\pi\ Z^2\ e^6}{3\ m_e c^3} \left(\frac{2\pi}{3 m_e k T}\right)^{1/2}\ \bar{g}_{\mathrm{ff}}(\nu,T)\ e^{-h\nu/kT}\ n_e\ n_{\mathrm{i}} \quad (6)$$

main frequency dependence is $j_\nu \propto e^{-h\nu/kT}$

$\rightarrow$ **flat spectrum**, cut off at $\nu \sim kT/h$

$\rightarrow$ can use to determine temperature of hot plasma (PS6)

integrated bremsstrahlung emission:

$$4\pi j_{\mathrm{ff}}(T) = 4\pi \int j_{\nu,\mathrm{ff}}(T)\ d\nu \quad (7)$$

$$= \frac{2^5\pi\ Z^2\ e^6}{3\ h m_e c^3} \left(\frac{2\pi k T}{3 m_e}\right)^{1/2}\ \bar{g}_{\mathrm{B}}(T)\ e^{-h\nu/kT}\ n_e\ n_{\mathrm{i}} \quad (8)$$

$$= 1.4 \times 10^{-27}\ \mathrm{erg\ s^{-1}\ cm^{-3}}\ \bar{g}_{\mathrm{B}}\ \left(\frac{T}{\mathrm{K}}\right)^{\frac{1}{2}} \left(\frac{n_e}{1\ \mathrm{cm}^{-3}}\right) \left(\frac{n_{\mathrm{i}}}{1\ \mathrm{cm}^{-3}}\right)$$

with $\bar{g}_{\mathrm{B}}(T) \sim 1.2 \pm 0.2$ a frequency-averaged Gaunt factor

Q: all of this was for emission–what about the thermal bremsstrahlung absorption coefficient?

Thermal Bremsstrahlung Absorption

for *thermal* system, *Kirchoff's law:* $S_\nu = B_\nu(T) = j_\nu/\alpha_\nu$

thus we have

$$\alpha_{\nu,\mathrm{ff}} = \frac{j_{\nu,\mathrm{ff}}}{B_\nu(T)} = \frac{4\ Z^2\ e^6}{3\ m_e hc}\left(\frac{2\pi}{3m_e kT}\right)^{1/2} \bar{g}_{\mathrm{ff}}(\nu,T)\ \nu^{-3}\ \left(1 - e^{-h\nu/kT}\right) n_e\, n_{\mathrm{i}}$$

note that $\alpha_{\nu,\mathrm{ff}} \propto n_e n_{\mathrm{i}}$: two factors of density!
contrasts with usual $\alpha_\nu = n\sigma_\nu \propto n$: *one* factor of density

Why? *brems. absorption is a 3-body process:* $\gamma + e^- +$ ion
photon absorption depends on both n_e and n_i

Bremsstrahlung Absorption: Frequency Dependence

$$\alpha_{\nu,\text{ff}}(T) = \frac{4\ Z^2\ e^6}{3\ m_e hc}\left(\frac{2\pi}{3m_e kT}\right)^{1/2}\ \bar{g}_{\text{ff}}(\nu, T)\ \nu^{-3}\ \left(1 - e^{-h\nu/kT}\right)\ n_e\ n_{\text{i}}$$

limits:

- $h\nu \ll kT$: $\alpha_{\nu,\text{ff}} \propto \nu^{-2}$
- $h\nu \gg kT$: $\alpha_{\nu,\text{ff}} \propto \nu^{-3}$

Q: sketch optical depth vs ν? implications?

Bremsstrahlung Self-Absorption

bremsstrahlung optical depth at small ν:

$$\tau_\nu \propto \alpha_{\nu,\mathrm{ff}} \propto \nu^{-2} \tag{9}$$

guaranteed optically thick below some ν
$\rightarrow$ free-free emission is absorbed inside plasma:
bremsstrahlung self-absorption

thus observed plasma spectra should have three regimes

- small ν: $\tau_\nu \gg 1$, optically thick, $I_\nu \rightarrow B_\nu \propto \nu^3$
- $h\nu < kT$: optically thin, $I_\nu \rightarrow j_\nu s$ *flat* vs ν
- $h\nu \gg kT$: thermally suppressed, $I_\nu \rightarrow j_\nu s \sim e^{-h\nu/kT}$

∞ *Q: expected X-ray count spectrum for galaxy cluster?*

`www: observations`

Bremsstrahlung and Stellar Interiors

in stellar interiors, bremsstrahlung known as **free-free** emission

stellar flux transfer depends on frequency-averaged opacity

Q: what's opacity?

Q: how do we perform frequency average?

Bremsstrahlung and Stellar Interiors

opacity defined via

$$\alpha_\nu = n\sigma_\nu = \rho\kappa_\nu \tag{10}$$

and in thermal radiation field, frequency avg is **Rosseland mean**

$$\frac{1}{\alpha_{\rm R}} = \frac{\int (\alpha_\nu + \varsigma_\nu)^{-1} \partial_T B_\nu \, d\nu}{\int \partial_T B_\nu \, d\nu} \tag{11}$$

for bremsstrahlung at $h\nu < kT$

$$\alpha_\nu \sim \frac{Z^2 \nu^2}{T^{3/2}} \, n_e \, n_i \tag{12}$$

Q: and so what do we expect for $\alpha_{\rm R}(\rho, T)$*?*

Free-Free Opacity

free-free = bremsstrahlung: $\alpha_\nu \sim \nu^2/T^{3/2}\ n_e\ n_i$

scalings in Rosseland mean for bremsstrahlung:

- $h\nu \sim kT$
- $n_i \propto \rho$, and $n_e \propto \rho/\mu_e$

gives **Kramer's Law** for opacity

$$\kappa_{\mathrm{ff}}(\rho, T) \sim \frac{\rho}{\mu_e T^{7/2}} \tag{13}$$

appears in local flux expression

$$F(z) = -\frac{16\sigma T^3}{3\kappa\rho}\frac{\partial T}{\partial z} \tag{14}$$

important for stellar interiors

bonus: same scaling works for bound-free!

Build Your Toolbox: Bremsstrahlung

microphysics: matter-radiation interactions

Q: physical origin of bremsstrahlung?

Q: physical nature of sources?

Q: spectrum characteristics?

Q: frequency range?

real/expected astrophysical sources of Thomson scattering

Q: where do we expect this to be important?

Q: relevant EM bands? temperatures?

Toolbox: Bremsstrahlung

emission physics

- physical origin: non-relativistic Coulomb acceleration in free electrons and ions interactions
- physical sources: need free e^- and ions $\rightarrow$ ionized gas= plasma
- spectrum: continuum emission
  for thermal plasma, $j_\nu \sim const$ vs ν
  up to *exponential cutoff* $e^{-h\nu/kT}$

astrophysical sources of bremsstrahlung emission/absorption

- sites are highly ionized gas: stellar interiors.
  hot nebulæ: accretion disks, Hii regions, supernova remnants, galaxy clusters. the early Universe.
- EM bands radio to X-ray
- temperatures hot enough to be ionized: $T \gtrsim 10^5$ K

Radiation and Bound States

Radiation and Bound States

In the course up until now, focused largely on
continuum radiation
i.e., processes that emit/absorb across a wide range of ν
and (mostly) involve *unbound* electrons

- blackbodies
- bremsstrahlung
- Thompson scattering

But what about *lines*!?
these arise from transitions involving electrons in *bound states*
$\rightarrow$ atoms and molecules

To discuss these, begin with refresher on atoms and molecules

Atomic Structure

Gossip Break: Wiesskopf's Pauli Story

Atomic Structure: Order of Magnitude

Atoms and molecules inherently *quantum* systems

Cowgirl/cowboy view of hydrogen:
quantum bound state of electron around proton, with energy

$$E = \frac{p^2}{2m_e} - \frac{e^2}{r} \tag{15}$$

but Heisenberg: $rp \geq \hbar/2$

Wild West: cowgirl/cowboy approximation $p \sim \hbar/r$

$$E = \frac{\hbar^2}{2m_e r^2} - \frac{e^2}{r} \tag{16}$$

Q: so how to find ground state?

Heisenberg-ized sketch of hydrogen energy

$$E = \frac{\hbar^2}{2m_e r^2} - \frac{e^2}{r} \tag{17}$$

ground state: E is *minimum*

$$\partial_r E = -\frac{\hbar^2}{m_e r^3} + \frac{e^2}{r^2} = 0 \tag{18}$$

gives electron radius $r_{\text{min}} \equiv a_0$:

$$a_0 = \frac{\hbar^2}{e^2 m_e} = 0.05 \text{ nm} \tag{19}$$

and electron energy $E(r_{\text{min}}) \equiv E_1$:

$$E_1 = -\frac{e^4 m_e}{2\hbar^2} = -\frac{1}{2}\alpha^2 m_e c^2 = 13.7 \text{ eV} \tag{20}$$

where $\alpha = e^2/\hbar c \approx 1/137$

Q: how do these compare with results of honest calculation?

Hydrogen Atom: Honest Non-Relativistic Results

non-relativistic Schrödinger ignores relativistic effects

- electron (and proton) spins absent from Hamiltonian
  $\rightarrow$ electron orbit properties independent of spin

for **hydrogen-like species:** single electron, nuclear charge Z

ground state properties

- energy $E_1 = -Z^2 e^4 m_e / 2\hbar^2$
- mean radius $\langle r_1 \rangle = a_0 / Z$
- electron expected speed $\langle v_1 \rangle = Ze^2/\hbar = Z\alpha c$
  so that $\beta_1 = v_1/c = Z\alpha \approx Z/137 \ll 1$ for most atoms
  if not: non-relativistic is bad approximation!

Q: what about excited states: how many?
Q: how do E_n, r_n, v_n vary with n?

excited states, ignoring spin effects (*non-relativistic*):
for each integer $n = 1, 2, 3, \ldots$

- $E_n = E_1/n^2$
- $\langle r \rangle_n = n^2 \ r_1$
- $\langle v \rangle_n = v_1/n$

Lessons

- H has an infinite "tower" of bound states
- $\langle r \rangle_n \propto n^2$: *principal quantum number* n controls *radial* part of wavefunction
- as $n \to \infty$: bigger radius, slower, more weakly bound

hydrogen wavefunction: 3-D system $\to$ need 3 quantum numbers
Q: what are the other two?

Non-relativistic hydrogen wavefunction: states specified by

- *principal* quantum number $n = 1, 2, \ldots$
  controls wavefunction dependence on r
- *orbital angular momentum* $\ell = 0, 1, \ldots, n-1$
  $\hat{L}^2 \psi = \ell(\ell+1)\hbar^2 \psi$
  controls wavefunction dependence on θ
- *z-projection* $\hat{L}_z \psi = m\hbar\psi$ with $m = -\ell, \ldots, +\ell$
  controls wavefunction dependence on ϕ

in non-relativistic case: energy only depends on n
all states with fixed n are *degenerate* (same energy)

- at each ℓ value: $2\ell + 1$ "substates" of different m
- each of which has 2 possible e spin states: $s_z = \pm 1/2$
- at each n: a total of $2\sum_{\ell=0}^{n-1} 2\ell + 1 = 2n^2$ states
  all with the same energy

Q: effect of full relativistic treatment?

Realistic Atoms

for *hydrogen*

Schrödinger: $v_n/c = \alpha/n \ll 1$, $|E_n| = \alpha m_e c^2/2 \ll m_e c^2$

$\rightarrow$ electron motion is (very) non-relativistic: approx justified!

$\rightarrow$ expect relativistic corrections to be small

Full relativity: Dirac equation

Hamiltonian includes spins of electron and proton

new interactions are $\propto \beta = v/c$ or β^2 $\rightarrow$ small corrections

$\rightarrow$ lifts degeneracy of levels at same n