Astronomy 501: Radiative Processes

Lecture 23

Oct 17, 2022

Announcements:

- **Problem Set 7 due Friday**
- Midterm exams almost graded; discussion today

Last time:

Annie Cannon and the mystery of stellar H lines

Q: who was Annie Cannon? what's the mystery?

Atoms in thermal equilibrium

Q: ratio n_2/n_1 *of number of atoms two states in thermal equilibrium?*

Annie Cannon and the Mystery of Stellar Hydrogen Lines

turn of 20th century: birth of stellar spectroscopy
stellar spectra classified according to spectral lines

master classifier: Annie Jump Cannon

later also determined stellar temperatures

hydrogen lines are prominent in some stars
but strange result: `www: data`

- H lines are weak for hottest stars
- H lines are weak for coldest stars
- H lines strongest for middle temperatures

Mystery: why this behavior?

Thermal Population of Atomic States

Boltzmann: consider a single atomic state having energy E_i for an ensemble of $n_{\rm tot}$ atoms in thermodynamic equilibrium at T

the population = numbers n_i of atoms in state i is

$$n_i = \frac{n_{\rm tot}}{Z} g_i e^{-E_i/kT} \tag{1}$$

with g_i the degeneracy of state i, and **partition function**

$$Z = \sum_{\text{states } i} e^{-E_i/kT} \tag{2}$$

and thus probability of state i is $p_i = g_i e^{-E_i/kT} / \sum_j g_j e^{-E_j/kT}$ with $\sum_i p_i = 1$

for two states of energies E_1, $E_2 > E_1$

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-(E_2 - E_1)/kT} \tag{3}$$

neutral hydrogen partition function, with $\beta = 1/kT$

$$Z(\mathrm{H}) = \sum g_n e^{-\beta E_n} = 2 \sum_{n=1}^{\infty} n^2 \; e^{\beta B/n^2} \tag{4}$$

$e^{\beta B/n^2} \to 1$ for large n, so

$$Z(\mathrm{H}) \approx 2 \sum_{\text{large } n}^{\infty} n^2 \sim n_{\text{max}}^3 \to \infty \tag{5}$$

infinite partition function!

but what does this mean?!
strictly: probability to be in state i is $p_i \propto 1/Z = 0$?!
that is: *high probability to be at high n*

physically: if H atoms in equilibrium with a thermal bath at T
and all states n are accessible
then eventually all atoms fluctuate to high $n \to$ *ionized!*

this can't be right! atoms do exist! *Q: what's the fix?*

Partition Function Cutoff

We implicitly assumed that we could carry our sum out to *arbitrarily large* n

While it is true that atomic H has such states
recall $r_n = n^2 a_0$: high-n states are physically large!

physically, real e orbits in an H atom cannot extend beyond the nearest-neighbor H atom
which typically lies at distance $d_{\rm max}$ such that $n_{\rm H} d^3 \sim 1$
or $d_{\rm max} \sim n_{\rm H}^{-1/3}$

setting $d_{\rm max} = n_{\rm max}^2 a_0$, we estimate

$$n_{\rm max} \sim \sqrt{d_{\rm max} a_0} \sim \left(a_0^3 n_{\rm H} \right)^{-1/6} \sim 10^4 \left(\frac{n_{\rm h}}{1 \ {\rm atom/cm^3}} \right)^{-1/6} \qquad (6)$$

but: a very Wild West estimate! real physics is more complex...

Thermodynamics of Ionization

consider a hydrogen gas in thermodynamic equilibrium at T
ionization and recombination both occur

$$\mathrm{H} + \gamma \leftrightarrow p + e \tag{7}$$

and the number densities n_e, n_p, and n_H adjust themselves until the recombination and ionization rates are equal

this equilibrium determines a relationship among the densities which we want to find

Method I (R&L):
starting point–ratio ions with free e at speed v
to neutral hydrogen atoms in the ground state

$$\frac{\delta n_+(v)}{n_\mathrm{H}} = \frac{\delta g(v)}{g_\mathrm{H}} e^{-[E_e(v)-E_1]/kT} = \frac{\delta g(v)}{g_\mathrm{H}} e^{-(B+m_e v^2/2)/kT} \tag{8}$$

where $B = -E_1 = 13.6$ eV is hydrogen binding energy

Boltzmann gives

$$\frac{\delta n_+(v)}{n_{\rm H}} = \frac{\delta g(v)}{g_{\rm H}} e^{-(B+m_e v^2/2)/kT} \tag{9}$$

and with statistical weight

$$\begin{aligned} g(v) &= g_p \; g_e && (10) \\ &= 2 g_p \frac{dx\,dy\,dz\;\; dp_x\,dp_y\,dp_z}{h^3} && (11) \end{aligned}$$

where volume element chosen to enclose one electron, so that number density $n_e = 1/d^3\vec{x} = 1/dxdydz$, and thus

$$\begin{aligned} \frac{n_p}{n_{\rm H}} &= \frac{4\pi}{h^3 n_e} \frac{g_p}{g_{\rm H}} \int e^{-(B+p_e^2/2m)/kT} \; p^2 \; dp && (12) \\ &= \frac{4\pi}{n_e} \frac{g_p}{g_{\rm H}} \left(\frac{2kT}{m_e h^2}\right)^{3/2} e^{-B/kT} \int_0^\infty e^{-x^2} x^2 \; dx && (13) \end{aligned}$$

and we arrive at the **Saha equation**

$$\frac{n_e n_p}{n_{\mathrm{H}}} = \frac{g_e g_p}{g_{\mathrm{H}}} \left(2\pi \frac{m_e m_p}{m_{\mathrm{H}}} \frac{kT}{h^2} \right)^{3/2} e^{-B_{\mathrm{H}}/kT} \tag{14}$$

where *hydrogen binding energy*

$B_{\mathrm{H}} = (m_e + m_p - m_{\mathrm{H}})c^2 = 13.6 \text{ eV}$

Q: behavior at high T? low T? does this make sense?

The Saha Equation

define *ionization fraction*

$$x_e = \frac{n_e}{n_{\text{tot}}} \quad (15)$$

with total electron number density $n_{\text{tot}} = n_e + n_{\text{H}}$
using $n_e = n_p$ (charge neutrality):

$$\frac{x_e^2}{1 - x_e} \approx \frac{2(2\pi m_e kT/h^2)^{3/2}}{n_{\text{tot}}} e^{-B_{\text{H}}/kT} = \frac{n_{\text{Q},e}}{n_{\text{tot}}} e^{-B_{\text{H}}/kT} \quad (16)$$

for $kT \gg B_{\text{H}}$, $x_e \to 1$: (nearly) fully ionized
for $kT \ll B_{\text{H}}$, $x_e \ll 1$: (nearly) fully neutral

but note that, e.g., temperature at which $x_e = 1/2$
also depends on particle density n_{tot}

Awesome Saha Example: Cosmic Recombination

the early universe: *hot!*
$kT \gg B_{\rm H} \rightarrow$ totally ionized, $x_e \rightarrow 1$

present-day universe: on average, *cold!*
$T = 2.725$ K $\rightarrow$ if no stars, U would be neutral, $x_e \rightarrow 0$

thus there was a transition: **(re)combination**
our mission: estimate $T_{\rm rec}$ = when cosmic $x_e = 1/2$

Q: naïve, zeroth order estimate?
Q: how to improve?

naïvely, expect recombination when $kT_{\rm rec} \sim B_{\rm H}$
with $B_{\rm H} = 13.6$ eV, this gives
$T_{\rm rec,naive} = B_{\rm H}/k \sim 120,000$ K

but we can do better using Saha
exponential dependence on $B_{\rm H}$, but also
dependence on $n_{\rm tot}$

big-bang nucleosynthesis teaches* us that
the cosmic baryon-to-photon ratio is

$$\eta \equiv \frac{n_{\rm b}}{n_\gamma} = 6 \times 10^{-10} \tag{17}$$

most baryons are hydrogen, so $n_{\rm tot} \sim n_{\rm b}$
and thus there are *many photons for each p and e*

Q: anticipated effect on $T_{\rm rec}$? higher or lower than $T_{\rm rec,naive}$?

*How? find out next semester in Physical Cosmology!

many photons per p and $e \to$ very easy to ionize H

- when $kT < B_{\mathrm{H}}$, there are still many photons in Wien tail with $h\nu > B_{\mathrm{H}}$
- thus expect $T_{\mathrm{rec}} < T_{\mathrm{rec,naive}}$

in detail:
recall that $n_\gamma \sim (kT/hc)^3$, so

$$n_{\mathrm{tot}} \sim \eta n_\gamma \sim \eta (kT/hc)^3 \tag{18}$$

and so Saha becomes

$$\frac{x_e^2}{1-x_e} \sim \frac{1}{\eta}\left(\frac{m_e c^2}{kT}\right)^{3/2} e^{-B_{\mathrm{H}}/kT} \tag{19}$$

note: $1/\eta \gg 1$ and $m_e c^2/kT \gg 1$

so when $x_e = 1/2$ we have (PS 7)
$T_{\mathrm{rec}} \simeq T_{\mathrm{rec,naive}}/40 \sim 3000$ K
k$T_{\mathrm{rec}} \simeq 0.3$ eV $\ll B_{\mathrm{H}}$
and thus $1 + z_{\mathrm{rec}} = T_{\mathrm{rec}}/T_0 \sim 1000$