# **Astronomy 501: Radiative Processes**

Lecture 3

Aug 26, 2022

Announcements:

- **TA Chris Tandoi** is here–take a bow!
- **Problem Set 1 posted on Canvas, due next Friday**
  you may speak to me, the TA, and other students
  but you must *understand* your own answers
  and write them *yourself* and *in your own words*
  you may *not* consult old 501 problem sets/solutions

- Please leave your camera on if at all possible

- next Monday and Wednesday: **meet in person!**
  *please please mask up!*

# Last Time

a blizzard of definitions! ...and more today

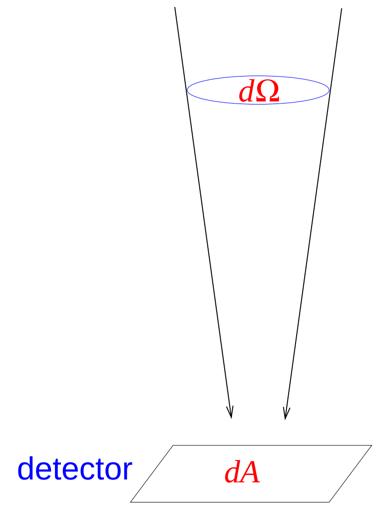
*Q: what is flux? specific flux? how are they different?*

*Q: what is flux in ordinary experience/language?*

*Q: what is intensity? how does it differ from flux?*

*Q: what is intensity in ordinary experience/language?*

# Intensity or Surface Brightness




Isolate small region (solid angle $d\Omega$) of sky
by introducing a *collimator*

*If* source is extended over this region sky,
energy flow received depends on
collimator acceptance $d\Omega$: $d\mathcal{E} \propto dA \; dt \; d\Omega$

so define flux per unit "surface area" of sky:
**intensity** or surface brightness (or sometimes just "brightness")

$$I = \frac{d\mathcal{E}}{dt \; dA \; d\Omega} \tag{1}$$

cgs units: $[I] = [\mathrm{erg\,cm^{-2}\,s^{-1}\,sr^{-1}}]$, with sr = steradian

*Q: what has been implicitly assumed?*

have assumed light travels in straight lines: "*rays*"

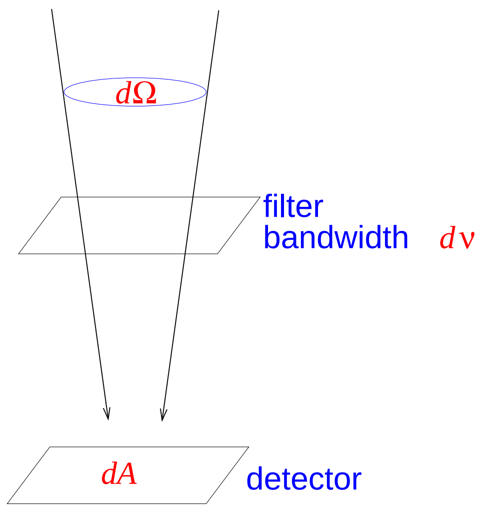
- for infinitesimal solid angle $d\Omega$, collimator selects a small "*bundle*" or "pencil" (Chandrasekhar) of rays
- intensity $I$ describes *one* individual ray (one direction) while flux describes *all* rays (all directions)

thus: implicitly adopted *geometric optics* approximation:
we have ignored diffraction effects
good as long as system scales $\gg \lambda$

Note: for each direction/ray $(\theta, \phi)$, intensity $I$ takes *single value*
resulting image is **grayscale map** of all-color brightness

*Q: What if we are interested in the spectrum?*

# Specific Intensity




introduce a filter or grating, to disperse by $\lambda$
energy received: $d\mathcal{E} \propto dA\ dt\ d\Omega\ d\nu$

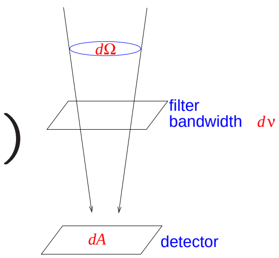
define **specific intensity** or **spectral energy distribution (SED)** or **spectral brightness**

$$I_\nu = \frac{d\mathcal{E}}{dt\ dA\ d\Omega\ d\nu} \tag{2}$$

cgs units: $[I_\nu] = [\mathrm{erg\,cm^{-2}\,s^{-1}\,sr^{-1}\,Hz^{-1}}]$

*Q: what is an image of $I_\nu$ at one $\nu$?*

For each $\nu$ and direction $(\theta, \phi)$: specific intensity is a *number*: the brightness of a ray at that sightline




$$I_\nu = I_\nu(\theta, \phi) \tag{3}$$

*at fixed $\nu$*: each image is a grayscale map showing monochromatic brightness over imaged region

*across different $\nu$:*

- $I_\nu$ images are *stack of maps at each $\nu$*
- each $I_\nu(\theta, \phi)$ gives spectrum of each ray!

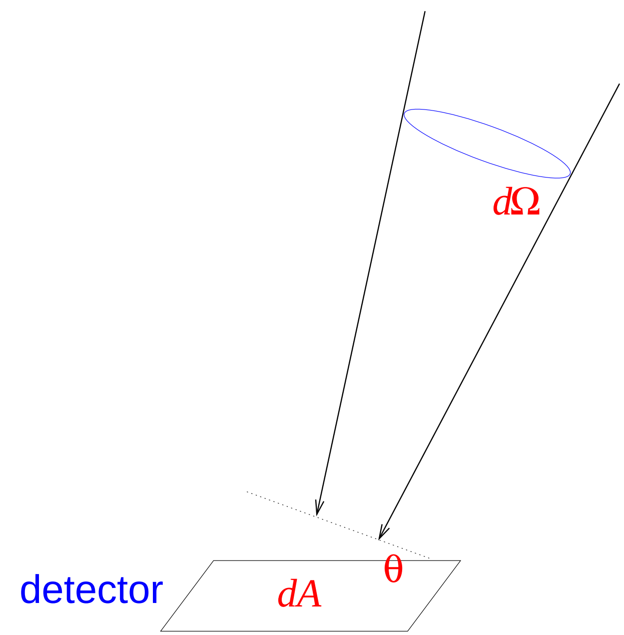
*maximal info about the source!* `www: examples`

- a less compact but more explicit notation is $dI/d\nu$
- **total intensity:** $\int I_\nu \, d\nu$
- can also define $I_\lambda = dI/d\lambda$ and $I_E = dI/dE$

# Intensity for Arbitrary Incidence

if incidence is at angle $\theta$ to normal
radiation "sees" detector with
projected normal area $dA_{\perp} = \cos\theta \, dA$

where unit direction vectors give
$\cos\theta = \hat{n}_{\rm rad} \cdot \hat{n}_{\rm detector}$




and thus energy flow onto detector has geometric factor

$$d\mathcal{E} = I_\nu \;\cos\theta \; dA \; d\Omega \; dt \; d\nu \tag{4}$$

*Q: so how to find specific flux?*

# Flux from Intensity

flux measures rays in all direction
passing through surface of normal (unit vector $\hat{n}_{\mathsf{surf}}$)

the contribution to the specific flux through area $dA$
from a ray with solid angle $d\Omega$,
coming from direction $\hat{n}_{\mathsf{ray}}$, is:

$$dF_\nu = I_\nu \ \hat{n}_{\mathsf{ray}} \cdot \hat{n}_{\mathsf{surf}} \ d\Omega = I_\nu \ \cos\theta \ d\Omega \tag{5}$$

and thus if we sum over all solid angles
get get *net* specific flux

$$F_\nu = \int I_\nu \ \cos\theta \ d\Omega \tag{6}$$

∞ Note: in general, $I_\nu$ varies in different directions
e.g., in spherical coords, $I_\nu(\theta, \phi)$
$\rightarrow$ an then integral for $F_\nu$ is non-trivial

# Mean Intensity

the direction-averaged **mean or average intensity** is

$$J_\nu = \langle I_\nu \rangle \tag{7}$$

$$= \frac{\int I_\nu \; d\Omega}{\int d\Omega} \tag{8}$$

$$= \frac{1}{4\pi} \int I_\nu \; d\Omega \tag{9}$$

note that this is a *scalar* average = undirected
so oppositely-directed rays do *not* cancel
( unlike *flux*, which has as associated direction (normal)

important special case:
*if* $I_\nu$ is *same* in all directions: *isotropic* *Q: then what is* $J_\nu$*?*



Consider isotropic radiation incident on two-sided detector
*Q: what's the net flux?*

# Isotropic Radiation

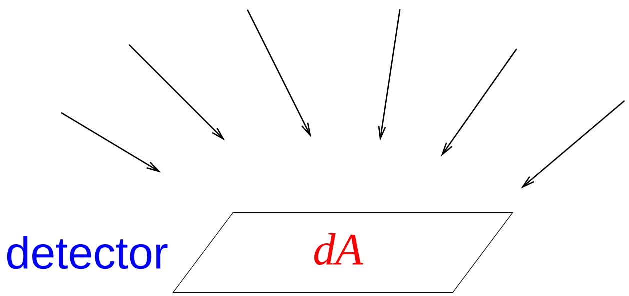
**isotropy:** $I_\nu(\theta,\phi) = I_\nu^{\rm iso}$ indept of $\theta$ and $\phi$
"looks the same in all directions"
for isotropic radiation: $J_\nu = I_\nu^{\rm iso}$

on *two-sided detector*, net flux sums over *all* solid angles

$$\begin{aligned} F_{\nu,2-{\rm sided}}^{\rm iso} &= I_\nu^{\rm iso} \int_{4\pi} \cos\theta \; d\Omega = I_\nu^{\rm iso} \int_0^{2\pi} d\phi \int_{-1}^{1} \cos\theta \; d\cos\theta \\ &= 2\pi \, I_\nu^{\rm iso} \int_{-1}^{1} \mu \; d\mu = \pi \, I_\nu^{\rm iso} \left[\mu^2\right]_{-1}^{+1} = 0 \end{aligned} \quad (10)$$

where in spherical coords $d\Omega = \sin\theta \; d\theta \; d\phi = d\cos\theta \; d\phi$
and where $\theta \in [0,\pi]$, $\cos\theta = \mu \in [-1,+1]$, and $\phi \in [0,2\pi]$

*Q: why physically is* $F_{\nu,2-{\rm sided}} = 0$*?*




on *one-sided detector = plane* $\theta \in (0, \pi/2)$:

$$F_{\nu,1-\mathrm{sided}}^{\mathrm{iso}} = \pi\, I_{\nu}^{\mathrm{iso}} \left[\mu^2\right]_0^{+1} = \pi\, I_{\nu}^{\mathrm{iso}} \tag{11}$$

a factor $\pi$ larger than naive Gauss' law result

*Q: why?*

# On Frequency and Wavelength

For most of the course, we will describe specific intensity using $I_\nu \equiv dI/d\nu$, i.e., in "frequency space"

But we could as well use $I_\lambda \equiv dI/d\lambda$: "wavelength space"

Of course, the two are related: in $(\nu, \nu + d\nu)$
the intensity $I_\nu\, d\nu$ is *equal to* $I_\lambda\, d\lambda$
where $(\lambda, \lambda + d\lambda)$ is the *corresponding* wavelength interval:
i.e., $\nu = c/\lambda$, and $d\nu = -c\, d\lambda/\lambda^2$

Thus the two intensity descriptions differ by a change of variable and thus by a Jacobian factor:

$$I_\lambda = \left|\frac{d\nu}{d\lambda}\right| \; I_\nu = \frac{c}{\lambda^2}\; I_{\nu(\lambda)} \qquad (12)$$

- same Jacobian factor needed for $F_\lambda = cF_\nu/\lambda^2$, and for $u_\lambda$, etc.
- note that $\lambda I_\lambda = \nu I_\nu$: both give the intensity per unit log interval $|d\lambda/\lambda| = |d\nu/\nu|$; good to show on plots!

# Photon Number

when using the photon picture of light
the basic units are *counts = number of photons*
where for monochromatic photons, $d\mathcal{E} = E_\nu \, dN = h\nu \, dN$

$\rightarrow$ useful to introduce the specific *number* intensity

$$\mathcal{I}_\nu = \frac{dN_\gamma}{dt \;\, dA \;\, d\Omega \;\, d\nu} = \frac{1}{h\nu} \; \frac{d\mathcal{E}}{dt \;\, dA \;\, d\Omega \;\, d\nu} = \frac{I_\nu}{h\nu} \tag{13}$$

and specific *number* flux

$$\Phi_\nu = \int \mathcal{I}_\nu \;\, \cos\theta \;\, d\Omega = \frac{1}{h\nu} \int I_\nu \;\, \cos\theta \;\, d\Omega = \frac{F_\nu}{h\nu} \tag{14}$$

# Specific Intensity Knows All!

Specific intensity $I_\nu(\theta, \phi)$ characterizes the photon population

- at each point of space
- accounting for rays in all directions
- across all frequencies

$I_\nu$ fully* describes the photon distribution in all space!

*Well, not yet polarization, but we'll add that later

From $I_\nu$ can infer other interesting properties of the photon photon population

Recall: photons carry *momentum* as well as energy

*Q: what determines photon momentum?*

*Q: how to calculate photon momentum flux?*

*Q: what is the physical significance of this flux?*

# Momentum Flux

consider the flux of photon *momentum*
in direction normal to area $dA$

For photons in solid angle $d\Omega$, from direction angle $\theta$
contribution to *number flux* is $d\Phi_\nu = I_\nu / h\nu \ \cos\theta \ d\Omega$

photon momentum $p_\nu = h\nu/c$ has normal component
$p_{\nu,\perp} = h\nu/c \ \cos\theta$

photon momentum flux $\perp$ surface is **radiation pressure**

$$P_\nu = \int p_{\nu,\perp} \ d\Phi_\nu = \frac{1}{c} \int I_\nu \ \cos^2\theta \ d\Omega \tag{15}$$

for *isotropic* radiation

$$P_\nu^{\rm iso} = 2\pi \frac{I_\nu^{\rm iso}}{c} \int_{-1}^{1} \mu^2 d\mu = \frac{4\pi}{3} \ \frac{I_\nu^{\rm iso}}{c} \tag{16}$$

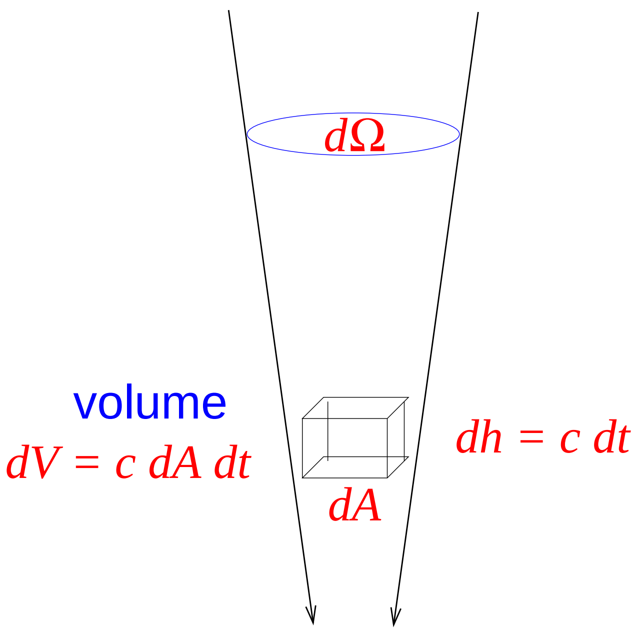
where $\mu = \cos\theta$

# Energy Density

consider a bundle of rays passing through a small volume $dV$

energy density $\upsilon_\nu(\Omega)$ for bundle defined by $d\mathcal{E} = \upsilon_\nu(\Omega)\, d\Omega\, dV$

but $dV = dA\, dh$, and flux thru height $dh$ in time $dt = dh/c$, so $dV = c\, dA\, dt$




thus we have

$$d\mathcal{E} = c\ \upsilon_\nu(\Omega)\ dA\ dt\ d\Omega \tag{17}$$

but by definition $d\mathcal{E} = I_\nu\ dA\ dt\ d\Omega$, so

$$\upsilon_\nu(\Omega) = \frac{I_\nu}{c} \tag{18}$$

specific energy density in bundle in solid angle $d\Omega$

$$\upsilon_\nu(\Omega) = \frac{I_\nu}{c} \tag{19}$$

so total energy density is

$$u_\nu = \int \upsilon_\nu \, d\Omega \tag{20}$$

$$= \frac{1}{c} \int I_\nu \, d\Omega \tag{21}$$

$$= \frac{4\pi J_\nu}{c} \tag{22}$$

we can similarly find the photon specific **number density**

$$n_\nu = \frac{u_\nu}{h\nu} = \frac{4\pi J_\nu}{hc\nu} \tag{23}$$

# Radiation Equation of State

recall: for isotropic radiation, pressure is momentum flux

$$P_\nu^{\rm iso} = \frac{4\pi}{3}\frac{I_\nu^{\rm iso}}{c} = \frac{u_\nu^{\rm iso}}{3} \tag{24}$$

pressure is 1/3 energy density, at each frequency!

note: relationship between pressure and (energy) density is an **equation of state**

thus people (=cosmologists) generalize this: $P = wu$
with $w$ the "equation of state parameter"
we find: for isotropic radiation, $w_{\rm rad} = 1/3$

# Integrated Intensity, Flux, Energy Density

specific intensity is per unit frequency: $I_\nu = dI/d\nu$
**total** or integrated intensity sums over all frequencies:

$$I = \int I_\nu \, d\nu \tag{25}$$

similarly, can define integrated flux

$$F = \int F_\nu \, d\nu \tag{26}$$

and integrate number and energy densities

$$n = \int n_\nu \, d\nu \tag{27}$$

$$u = \int u_\nu \, d\nu \tag{28}$$

*Q: what if we use a broadband filter? Examples? Why useful?*

# Cosmic Color Wheel: Filtered Light

we measure using a broadband filter (“color”)
with a finite passband window

write *transmission fraction or probability* for photons at $\nu$

$$W(\nu) = \left( \frac{\text{transmitted light}}{\text{incident light}} \right)_\nu \in [0, 1] \tag{29}$$

e.g., the classic $UBVGRIZ$..., or $ugrizY$

*Q: who uses these?* **`www: transmission curves`**

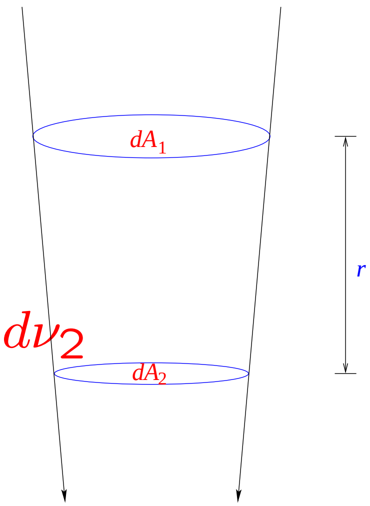
Then for band $i$, can define intensity

$$I_i = \int_{\text{band } i} I_\nu \, d\nu = \int W_i(\nu) \, I_\nu \, d\nu \tag{30}$$

and similarly for color flux $F_i = \int W_i(\nu) \, F_\nu \, d\nu$

# Constancy of Specific Intensity in Free Space

in free space: no emission, absorption, scattering,
consider rays normal to areas $dA_1$ and $dA_2$
separated by a distance $r$




energy flow is conserved, so

$$d\mathcal{E}_1 = I_{\nu_1}\ dA_1\ dt\ d\Omega_1\ d\nu_1 = d\mathcal{E}_2 = I_{\nu_2}\ dA_2\ dt\ d\Omega_2\ d\nu_2$$

- as seen by $dA_1$, the solid angle $d\Omega_1$ subtended by $dA_2$ is $d\Omega_1 = dA_2/r^2$, and similarly $d\Omega_2 = dA_1/r^2$
- and in free space $d\nu_1 = d\nu_2$, so:

$$I_{\nu_1} = I_{\nu_2} \tag{31}$$

$$I_{\nu_1} = I_{\nu_2} \tag{32}$$

thus: in free space, the intensity is constant along a ray
that is: intensity of an object in free space
is *the same* anywhere along the ray

so along a ray in free space: $I_\nu =$ constant
or along small increment $ds$ of the ray's path

$$\frac{dI_\nu}{ds} \stackrel{\text{free}}{=} 0 \tag{33}$$

this means: when viewing an object across free space,
the *intensity of the object is constant*
*regardless of distance to the object!*
⇒ **conservation of surface brightness**

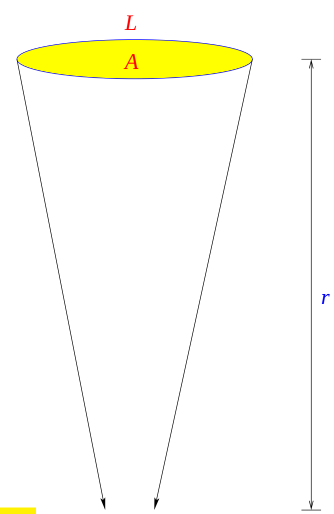
this is huge! and very useful!
*Q: what is implied? how can this be true–what about inverse square law? everyday examples?*

# Conservation of Surface Brightness

consider object in free space at distance $r$
with luminosity $L$ and project area $A \perp$ to sightline

*flux* from source follows usual inverse square

$$F = \frac{L}{4\pi r^2} \tag{34}$$




but *intensity* is flux *per solid angle*
and since $\Omega = A/r^2$, we have

$$I = \frac{F}{\Omega} = \frac{L/4\pi r^2}{A/r^2} = \frac{L}{4\pi A} \tag{35}$$

**surface brightness is independent of distance!**

and note $I = L/4\pi A$: intensity really is surface brightness
i.e., brightness per unit surface area and solid angle

# Consequences of Surface Brightness Conservation

*resolved* objects in free space
have *same $I$* at all distances

- Sun's brightness at surface is same as you see in sky
  but at surface subtends $2\pi$ steradian – yikes!

- similar planetary nebulae or galaxies all have similar $I$
  regardless of distance

- people and objects across the room don't look $1/r^2$ dimmer
  than things next to you
  fun exercise: when in your everyday life
  do you actually experience the inverse square law for flux?

# Adding Sources

matter can act as source and as sink for propagating light

the light energy added by glowing **source** in small volume $dV$, into a solid angle $d\Omega$, during time interval $dt$, and in frequency band $(\nu, \nu + d\nu)$, is written

$$d\mathcal{E}_{\text{emit}} = j_\nu \; dV \; dt \; d\Omega \; d\nu \tag{36}$$

defines the **emission coefficient**

$$j_\nu = \frac{d\mathcal{E}_{\text{emit}}}{dV \; dt \; d\Omega \; d\nu} \tag{37}$$

- power emitted per unit volume, frequency, and solid angle
- cgs units: $[j_\nu] = [\text{erg cm}^{-3} \text{ s}^{-1} \text{ sr}^{-1} \text{ Hz}^{-1}]$
- similarly can define $j_\lambda$, and integrated $j = \int j_\nu d\nu$

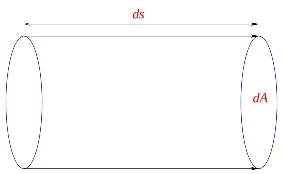
for *isotropic* emitters,
or for distribution of randomly oriented emitters, write

$$j_\nu = \frac{q_\nu}{4\pi} \tag{38}$$

where $q_\nu$ is radiated power per unit volume and frequency

sometimes also define *emissivity* $\epsilon_\nu = q_\nu / \rho$
energy emitted per unit freq and mass, with $\rho$ =mass density

beam of area $dA$ going distance $ds$
has volume $dV = dA\ ds$




so the *energy change* is $d\mathcal{E} = j_\nu\ ds\ dA\ dt\ d\Omega\ d\nu$
and the *intensity change* is

$$dI_\nu \overset{\text{sources}}{=} j_\nu\ ds \tag{39}$$

# Adding Sinks

as light passes through matter, energy can also be lost due to scattering and/or absorption

we *model* this as follows:

$$dI_\nu = -\alpha_\nu \ I_\nu \ ds \tag{40}$$

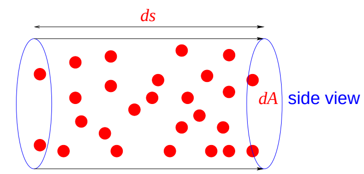
features/assumptions:

- losses proportional to distance $ds$ traveled
  *Q: why is this reasonable?*
- losses proportional to intensity
  *Q: why is this reasonable?*
- defines energy loss per unit pathlength, i.e.,
  **absorption coefficient** $\alpha_\nu$
  *Q: units/dimensions of* $\alpha_\nu$*?*

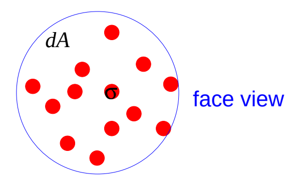
# Absorption Cross Section

consider "absorbers" with a number density $n_{\rm a}$
each of which presents the beam with an
effective *cross-sectional area* $\sigma_\nu$

over length $ds$, number of absorbers is
$dN_{\rm a} = n_{\rm a} \, dA \, ds$




a "dartboard problem"– over beam area $dA$
total "bullseye" area: $\sigma_\nu dN_{\rm a} = n_{\rm a}\sigma_\nu \, dA \, ds$




so absorption *probability* is

$$dP_{\rm abs} = \frac{\text{total bullseye area}}{\text{total beam area}} = n_{\rm a} \; \sigma_\nu \; ds \qquad (41)$$

*Q: for what length* $ds$ *does* $P_{\rm abs} \to 1$*?*
*Q: physical significance of* $n_{\rm a}\sigma_\nu$*?*

## Cross Sections, Mean Free Path, and Absorption

absorption probability large when photon travels **mean free path**

$$\ell_{\rm mfp} = \frac{1}{n_{\rm a}\sigma_\nu} \tag{42}$$

so we can write $dP_{\rm abs} = ds/\ell_{\rm mfp}$

and thus beam energy change is

$$d\mathcal{E} = -dP_{\rm abs}\mathcal{E} = -n_{\rm a}\sigma_\nu I_\nu \ ds \ dA \ dt \ d\Omega \ d\nu \tag{43}$$

which must lead to an intensity change

$$dI_\nu \overset{\rm abs}{=} -n_{\rm a} \ \sigma_\nu \ I_\nu \ ds \tag{44}$$

*Q: and so?*

$$dI_\nu \stackrel{\text{abs}}{=} -n_{\text{a}} \; \sigma_\nu \; I_\nu \; ds \tag{45}$$

has the expected form, and we identify the absorption coefficient

$$\alpha_\nu = n_{\text{a}} \; \sigma_\nu = \frac{1}{\ell_{\text{mfp}}} \tag{46}$$

note that absorption depends on

- *microphysics* via the cross section $\sigma_\nu$
- *astrophysics* via density $n_{\text{abs}}$ of scatterers

often, write $\alpha_\nu = \rho \kappa_\nu$,
defines **opacity** $\kappa_\nu = (n/\rho)\sigma_\nu \equiv \sigma_\nu/m$
with $m = \rho/n$ the mean mass per absorber



*Q: so what determines $\sigma_\nu$? e.g., for electrons?*

# Cross Sections

Note that the absorption **cross section** $\sigma_\nu$ is
and *effective* area presented by absorber

for "billiard balls" = neutral, opaque, macroscopic objects
this is the same as the geometric size
but generally, cross section is *unrelated to geometric size*
e.g., electrons are point particles (?) but still scatter light

- so *generalize* our ideas so that
  $dI_\nu = -n_{\rm a}\ \sigma_\nu\ ds$ *defines* the cross section
- determined by the details of light-matter interactions
- can be–and usually is!–frequency dependent
- differ according to physical process
  the study of which will be the bulk of this course!

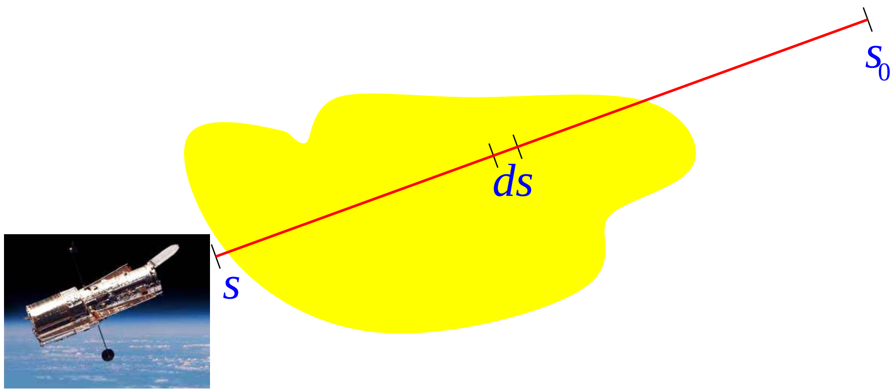
Note: "absorption" here is anything removing energy from beam
$\rightarrow$ can be true absorption, but also scattering

# The Equation of Radiative Transfer

Now combine effects of sources and sinks
that change intensity as light propagates

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu \qquad (47)$$

**equation of radiative transfer**




- fundamental equation in this course
- *sources* parameterized via $j_\nu$
- *sinks* parameterized via $\alpha_\nu = n_{\mathrm{a}}\,\sigma_\nu = \rho\kappa_\nu$

# Transfer Equation: Limiting Cases

equation of radiative transfer:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu \tag{48}$$

## Sources but no Sinks

if sources exist (and are independent of $I_\nu$) but no sinks: $\alpha_\nu = 0$

$$\frac{dI_\nu}{ds} = j_\nu \tag{49}$$

solve along path starting at sightline distance $s_0$:

$$I_\nu(s) = I_\nu(s_0) + \int_{s_0}^{s} j_\nu \, ds' \tag{50}$$

- the *increment* in intensity is due to integral of sources *along sightline*
- for $j_\nu \to 0$: free space case and $I_\nu(s) = I_\nu(s_0)$: recover surface brightness conservation!

# Special Case: Sinks but no Sources

if absorption only, no sources: $j_\nu = 0$

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu \tag{51}$$

and so on a sightline from $s_0$ to $s$

$$I_\nu(s) = I_\nu(s_0)\ e^{-\int_{s_0}^{s} \alpha_\nu\ ds'} \tag{52}$$

- intensity *decrement* is *exponential*!
- exponent depends on line integral of absorption coefficient

useful to define **optical depth** via $d\tau_\nu \equiv \alpha_\nu\ ds$

$$\tau_\nu(s) = \int_{s_0}^{s} \alpha_\nu\ ds' \tag{53}$$

and thus *for absorption only* $I_\nu(s) = I_\nu(s_0) e^{-\tau_\nu(s)}$