Astronomy 501: Radiative Processes

Lecture 36

Nov 18, 2022

Announcements:

- **Problem Set 11–final one!–due Friday**
  Q1 wordy but not much to calculate!
- Office Hours after class today or by appointment
- **Radiative Meme Submission on Canvas**
- If you missed Iben Lecture 2021: Vicki Kalogera, Illinois PhD
  "From Stars to Einstein's Waves" Friday 1pm webinar
- Early this morning: Artemis I launch and translunar injection

last time:

- cosmic rays *Q: what are they?*
  `www: cosmic rays spectra`

Cosmic Ray Propagation

consider cosmic ray protons and electrons
moving through interstellar space

Q: what interactions will each have?

Q: what will be the effect on interstellar matter?

Q: how will this affect CR propagation ("radiative transfer")?

Q: how could we detect evidence for this?

cosmic rays are highly energetic and penetrating
fill the Galaxy (and other galaxies)

cosmic ray electrons and protons
can and do collide with and scatter off interstellar matter
...and interstellar radiation!

- source of heat and ionization for interstellar matter
- propagation should include scattering effect

But more important:
Cosmic rays are **charged particles**
$\rightarrow$ couple to Galactic (and intergalactic!) magnetic fields

- cosmic ray trajectories are bent by fields
- *cosmic rays do not point back to their sources*
- *accelerated motion* means that *cosmic rays radiate*
- electrons much more strongly accelerated $\rightarrow$ e synchrotron radiation dominates

`www: radio continuum sky, edge-on spirals, SN remnants`

Synchrotron Radiation

Relativistic Motion in a Uniform B Field

Consider a relativistic classical particle, mass m, charge q moving in a uniform magnetic field $\vec{B}$ with no electric field $\vec{\mathcal{E}} = 0$

Equations of motion: total relativistic *energy* $E = \gamma mc^2$

$$\frac{dE}{dt} = mc^2 \frac{d\gamma}{dt} = q \ \vec{v} \cdot \vec{\mathcal{E}} = 0 \tag{1}$$

and so γ is *constant* and hence $|\vec{v}|$ is too

Equations of motion: *momentum*

$$\frac{d\vec{p}}{dt} = m \frac{d}{dt} \gamma \vec{v} = \frac{q}{c} \vec{v} \times \vec{B} \tag{2}$$

$$\frac{d}{dt}\gamma\vec{v} = \frac{q}{mc}\vec{v} \times \vec{B} \tag{3}$$

but γ and $|\vec{v}|$ are constant, so

$$\frac{d}{dt}\vec{v} = \frac{q}{\gamma mc}\vec{v} \times \vec{B} \tag{4}$$

take dot product with $\vec{B}$

$$\vec{B} \cdot \frac{d}{dt}\vec{v} = B\frac{d}{dt}v_{\parallel} = 0 \tag{5}$$

$\rightarrow$ *velocity component $v_{\parallel}$ parallel to $\vec{B}$ is constant*

decompose velocity into $\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp}$

$$\frac{d}{dt}\vec{v}_{\perp} = \frac{q}{\gamma mc}\vec{v}_{\perp} \times \vec{B} \tag{6}$$

Q: resulting motion orthogonal to field?

$$\frac{d}{dt}\vec{v}_\perp = \frac{q}{\gamma mc}\vec{v}_\perp \times \vec{B} = \vec{v}_\perp \times \vec{\omega}_B \tag{7}$$

perpendicular velocity *precesses* around $\vec{B}$
with gyrofrequency

$$\vec{\omega}_B = \frac{q}{\gamma mc}\vec{B} \tag{8}$$

note: nonrelativistic gyrofrequency $\omega_{B,\mathrm{nr}} = qB/mc$
is independent of v
but in relativistic case has factor $1/\gamma$

Q: full motion of charge?

orthogonal to $\vec{B}$, particle with speed $v_\perp$
moves in circle with **gyroradius**

$$r_{\rm g} = \frac{v_\perp}{\omega_B} = \frac{mc\gamma v_\perp}{qB} = \frac{cp_\perp}{qB} \tag{9}$$

thus the general motion is a combination of

- *constant velocity* $v_\parallel$ *along* $\vec{B}$
- *uniform circular motion in plane orthogonal* $\vec{B}$

net result: **spiral** around $\vec{B}$

numerically: gyroradius

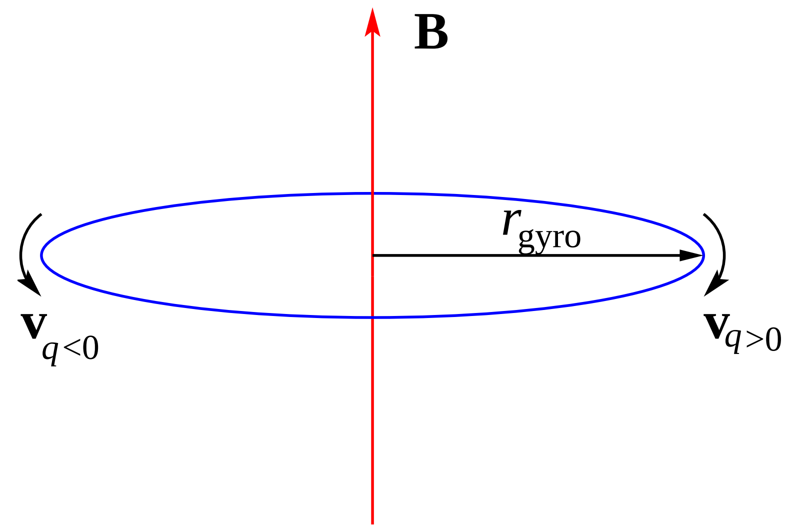
$$r_{\rm g} = 3.3 \times 10^{12}\ \text{cm} \left(\frac{cp_\perp}{1\ \text{GeV}} \right) \left(\frac{1\ \mu\text{Gauss}}{B} \right) \tag{10}$$

∞

Q: why these choices for $p_\perp$ *and* B*? implications?*
Q: what if p *very very large?*

charged particle in uniform $\vec{B}$

$$\begin{aligned} v_{\parallel} &= \text{const} \\ \frac{d\vec{v_{\perp}}}{dt} &= \vec{v} \times \vec{\omega}_B \\ v^2 &= v_{\parallel}^2 + v_{\perp}^2 = \text{const} \\ \gamma &= \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{E_{\text{tot}}}{mc^2} = \text{const} \end{aligned}$$



- *uniform velocity $v_{\parallel}$ along $\hat{B}$*
- *uniform circular motion orthogonal to $\hat{B}$*
  gyrofrequency $\omega_B = qB/\gamma mc$
  gyroradius $r_{\text{gyro}} = v_{\perp}/\omega_B = mc\gamma v_{\perp}/qB = cp_{\perp}/qB$
- net motion: **spiral around field line**

curved path $\rightarrow$acceleration $\rightarrow$radiation!

- non-relativistic particles: cyclotron radiation
- ultra-relativistic particles: **synchrotron radiation**

typical "blue collar" cosmic ray energy $E \sim cp \sim 1\mathrm{GeV}$
and typical interstellar magnetic field $B \sim 1\ \mu\mathrm{Gauss}$

thus typical cosmic ray gyroradius is

$$r_{\mathrm{g}} \sim 0.02\ \mathrm{AU} = 10^{-6}\ \mathrm{pc} \tag{11}$$

so $r_{\mathrm{g}} \ll$ solar system, interstellar scales
cosmic rays definitely do not move in straight lines

possible exception: $r_{\mathrm{g}} \gtrsim R_{\mathrm{MW}} \sim 10$ kpc
for $p \gtrsim 10^{10}$ GeV $= 10^{19}$ eV
$\rightarrow$ "ultra-high-energy cosmic rays"
www: arrival directions for UHECR

returning to typical cosmic rays: gyrofrequency

$$\nu_{\mathrm{g}} = \frac{\omega_{\mathrm{g}}}{2\pi} = \frac{eB}{2\pi\gamma mc} = 2.8\mathrm{Hz}\ \gamma^{-1}\ \left(\frac{B}{1\ \mu\mathrm{Gauss}}\right)\ \left(\frac{m_e}{m}\right) \tag{12}$$

Q: implications for electrons? protons?

gyrofrequency for mildly relativistic electrons:
cyclotron frequency $\nu_{\rm g} \sim few$ Hz
$\rightarrow$ very slow! huge wavelengths
if radiation is only at this frequency
would seem undetectable

but we will see: for relativistic electrons
radiation is at much higher frequencies!
synchrotron radiation

even so, low gyrofrequency hints that *radio* frequencies
likely to be important for synchrotron emission
`www: Kepler supernova remnant at 6 cm (VLA)`
Q: implications of intensity pattern?

Q: how to evaluate emitted synchrotron power from CR electrons?

Power Emitted by a Relativistic Charge

non-relativistic Larmor: $P' = 2q^2/3c^3 \; |\vec{a}'|^2$
want to re-express using 4-acceleration

can show: in instantaneous rest frame, $a^{0\prime} = 0$
and thus $|\vec{a}'|^2 = a \cdot a$
Lorentz-invariant Larmor expression for total radiated power

$$P = \frac{2}{3}\frac{q^2}{c^3} a \cdot a \qquad (13)$$

manifestly invariant, can evaluate in any frame

$$P = \frac{2}{3}\frac{q^2}{c^3} a \cdot a \tag{14}$$

in instantaneous rest frame, 4-acceleration transforms as

$$\begin{aligned} a'_{\|} &= \gamma^3 a_{\|} && (15) \\ a'_{\perp} &= \gamma^2 a_{\perp} && (16) \\ & && (17) \end{aligned}$$

and so power emitted is

$$\begin{aligned} P &= \frac{2}{3}\frac{q^2}{c^3} a' \cdot a' = \frac{2}{3}\frac{q^2}{c^3}(a'^2_{\perp} + a'^2_{\|}) && (18) \\ &= \frac{2}{3}\frac{q^2}{c^3}\gamma^4 \left(a^2_{\perp} + \gamma^2 a^2_{\|}\right) && (19) \end{aligned}$$

note large boost for relativistic particles ($P \propto \gamma^4$ or γ^6)

Synchrotron Radiation: Total Power

for isotropic electron population
average emitted power per electron:

$$P_e = \left|\frac{dE_e}{dt}\right| = \left(\frac{2}{3}\right)^2 r_0^2\ c\ \gamma^2 \beta B^2 = \frac{4}{3}\sigma_{\mathsf{T}}\ c\ \beta^2\gamma^2\ u_B \tag{20}$$

where $\sigma_{\mathsf{T}} = 8\pi r_0^2/3$ and $u_B = B^2/8\pi$

Q: energy dependence for non-relativistic electrons?
Q: energy dependence for ultra-relativistic electrons?
Q: stopping timescale for ultra-relativistic electrons?

Awesome Example: Radio Galaxies

awesome astrophysical example: radio galaxies

Q: what are they?

`www: radio images of Cygnus A, Centaurus A`

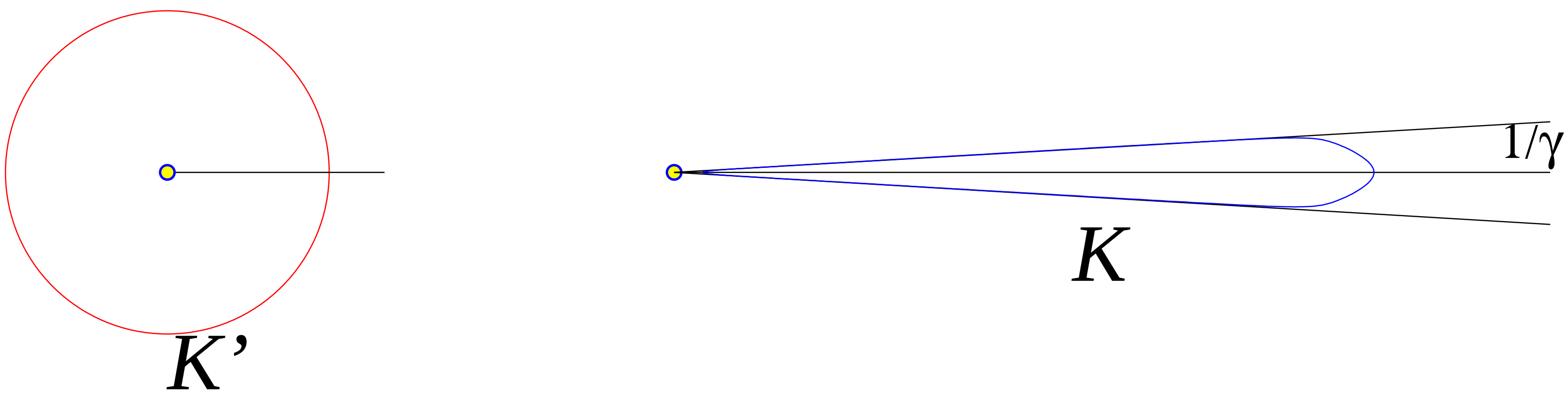
Q: how to find the spectrum of synchrotron radiation?

Q: why is it non-trivial? hint–think of relativistic circular motion

Spectrum of Synchrotron Radiation: Order of Magnitude

key issue:

radiation from a relativistic accelerated particle is *beamed* into forward cone of opening angle $\theta_{\text{beam}} \sim 1/\gamma$

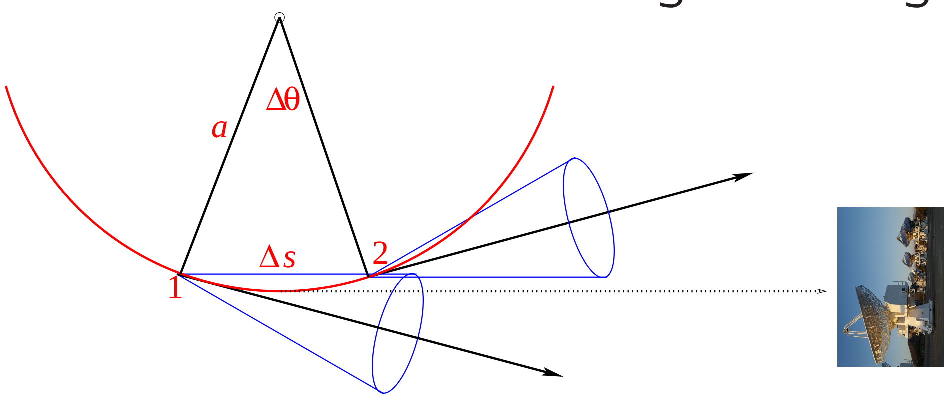


so observer receives pulses or "flashes" of radiation

spread over narrow timescale $\ll 2\pi/\omega_B$

sharply peaked signal in time domain

$\Rightarrow$ *broad signal in frequency domain* consider relativistic charge moving in circle of radius a



observer only sees emission over angular range

$$\Delta\theta \simeq 2\theta_{\rm beam} \simeq \frac{2}{\gamma} \tag{21}$$

representing a path length

$$\Delta s = a\ \Delta\theta = \frac{2a}{\gamma} \tag{22}$$

curvature radius $a = v/\omega_B \sin\alpha$, with $\sin\alpha = v_\perp/v$ so

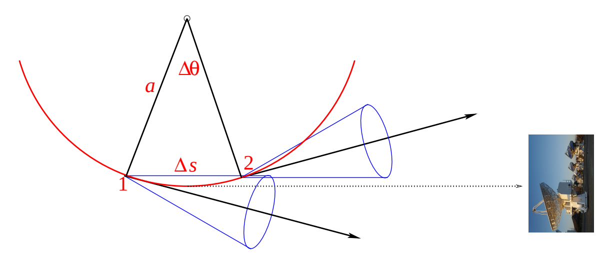
$$\Delta s \simeq \frac{2v}{\gamma\omega_B \sin\alpha} \tag{23}$$

if the particle *passes point 1* at t_1 and point 2 at t_2
$\Delta s = v(t_2 - t_1)$, and

$$\Delta t = t_2 - t_1 \simeq \frac{2}{\gamma\omega_B \sin\alpha} \tag{24}$$

what is *arrival time* of radiation?
note that point 2 is closer than point 1 by $\approx \Delta s$

$$\begin{aligned} \Delta t^{\mathrm{arr}} = t_2^{\mathrm{arr}} - t_1^{\mathrm{arr}} &= \Delta t - \frac{\Delta s}{c} \\ &= \Delta t \left(1 - \frac{v}{c}\right) \\ &= \frac{2}{\gamma\omega_B \sin\alpha}\left(1 - \frac{v}{c}\right) \end{aligned}$$



radiation arrive time duration

$$\Delta t^{\text{arr}} = \frac{2}{\gamma \omega_B \sin\alpha} \left(1 - \frac{v}{c}\right) \tag{25}$$

but note that $1 - v/c \approx 1/2\gamma^2$ for relativistic motion *Q:why?*

and thus radiation arrives in pulse of duration

$$\Delta t^{\text{arr}} \approx \frac{1}{\gamma^3 \omega_B \sin\alpha} \tag{26}$$

shorter than ω_B^{-1} by factor γ^3!

define **critical frequency**

$$\omega_{\text{c}} \equiv \frac{3}{2}\gamma^3 \omega_B \sin\alpha = \frac{3}{2}\gamma^2 \frac{qB\sin\alpha}{mc} = \frac{3}{2}\gamma^2\ \omega_{\text{g}} \sin\alpha \tag{27}$$

$$\nu_{\text{c}} = \frac{\omega_{\text{c}}}{2\pi} = \frac{3}{4\pi}\gamma^3 \omega_B \sin\alpha \tag{28}$$

Q: will radiation spectrum cut off above or below ω_{c}?

critical frequency

$$\nu_{\rm c} = \frac{3}{4\pi}\gamma^3 \omega_B \sin\alpha \sim \frac{1}{\Delta t^{\rm arr}} \tag{29}$$

Fourier transform of pulse $\Delta t^{\rm arr}$ broad up to $\nu_{\rm c}$ and should cut off above this

numerically:

$$\nu_{\rm c} = 25 \text{ MHz} \left(\frac{E_e}{1 \text{ GeV}}\right)^2 \left(\frac{B}{1\ \mu\text{Gauss}}\right) \sin\alpha \tag{30}$$

Q: lessons? irony?

critical = characteristic frequency $\nu_c \sim 25$ MHz $(E_e/1 \text{ GeV})^2$
typical cosmic-ray electrons emit in the observable *radio*
$\rightarrow$ *high-energy* electrons can emit *low-frequency* radiation!

expect synchrotron power of form $P(\omega) \sim P/\omega_{\rm C}\ F(\omega/\omega_{\rm C})$
with dimensionless function $F(x)$
- should be peaked at $x \sim 1$, then drop sharply
- can only be gotten from an honest calculation!

note: $P \propto \gamma^2$ but $\omega_{\rm C} \propto \gamma^2 \rightarrow P/\omega_{\rm C}$ indep of γ

for a particle with a fixed v and γ,
conventional to define synchrotron spectrum as

$$\frac{dP}{d\omega} = P(\omega) = \frac{\sqrt{3}}{2\pi}\frac{q^3 B \sin\alpha}{mc^2}\ F\left(\frac{\omega}{\omega_{\rm c}}\right) \tag{31}$$

with $\omega_{\rm c} \propto \gamma^2$

where the *synchrotron function* (derived in RL) is

$$F(x) = x\int_x^\infty K_{5/3}(t)\ dt \longrightarrow \begin{cases} \frac{4\pi}{\sqrt{3}\Gamma(1/3)}\left(\frac{x}{2}\right)^{1/3} & x \ll 1 \\ \left(\frac{\pi}{2}\right)^{1/2} e^{-x} x^{1/2} & x \gg 1 \end{cases} \tag{32}$$

with $K_{5/3}(x)$ the modified Bessel function of order 5/3
$\rightarrow$ *sharply peaked* at $\omega_{\rm max} = x_{\rm max}\omega_{\rm c} = 0.29\omega_{\rm c}$
`www: plot of synchrotron function`

Q: so is this the spectrum we would see for real CR es?

for a **single** electron γ
emission spectrum is synchrotron function $F(\omega/\omega_{\mathrm{c}})$
sharply peaked near $\omega_{\mathrm{c}} \propto \omega_{\mathrm{g}}\gamma^2$

but the *population* of cosmic-ray electrons
has a *spectrum* of energies and thus of γ

resulting synchrotron spectrum is

- *superposition* of peaks $\propto \gamma^2$,
- *weighted by electron energy spectrum*

Q: what if CRs had two energies? N energies?

Q: what does the real spectrum look like?

Q: what's the synchrotron spectral shape for the ensemble of all electron energies?

recall: cosmic-ray electron spectrum well-fit by *power law*
so number of particles with energy in $(E, E + dE)$ is

$$N(E)\ dE = C\ E^{-p}\ dE \tag{33}$$

and so

$$N(\gamma)\ d\gamma = C'\ \gamma^{-p}\ d\gamma \tag{34}$$

note that for a single electron v and γ
$P(\omega) \propto F(\omega/\omega_{\rm c})$ and $\omega_{\rm c} = \omega_{\rm g}\gamma^2$

so integrating over full CR spectrum means

$$\begin{aligned} \langle P(\omega)\rangle &= \int P(\omega)\ N(\gamma)\ d\gamma && (35) \\ &= C' \int P(\omega)\ \gamma^{-p}\ d\gamma && (36) \\ &\propto \int F\left(\frac{\omega}{\omega_{\rm g}\gamma^2}\right)\ \gamma^{-p}\ d\gamma && (37) \end{aligned}$$

Q: strategy?

$$\langle P(\omega)\rangle \propto \int F\left(\frac{\omega}{\omega_{\rm g}\gamma^2}\right)\ \gamma^{-p}\ d\gamma \tag{38}$$

change integration variable to $x = \omega/\omega_{\rm c} = \gamma^{-2}\omega/\omega_{\rm g}$
$\rightarrow \gamma = (\omega x/\omega_{\rm g})^{-1/2}$, and $d\gamma = -(\omega/\omega_{\rm g})^{-1/2}x^{-3/2}dx$

$$\langle P(\omega)\rangle \propto \left(\frac{\omega}{\omega_{\rm g}}\right)^{-(p-1)/2} \int F(x)\ x^{(p-3)/2}\ dx \tag{39}$$

and so

$$\langle P(\omega)\rangle \propto \omega^{-(p-1)/2} = \omega^{-s} \tag{40}$$

with **spectral index** $s = (p-1)/2$

even though each electron energy $\rightarrow$ peaked emission
average over power-law electron distribution
$\rightarrow$ power-law synchrotron emission

full expression for power-law electron spectrum of the form $dN/d\gamma = C\gamma^{-p}$

$$4\pi j_{\rm tot}(\omega) = \frac{\sqrt{3} q^3 C B \sin\alpha}{2(p+1)\pi m c^2} \, \Gamma\left(\frac{p}{4} + \frac{9}{12}\right) \, \Gamma\left(\frac{p}{4} - \frac{1}{12}\right) \, \left(\frac{m c \omega}{3 q B \sin\alpha}\right)^{-(p-1)/2} \tag{41}$$

with $\Gamma(x)$ the gamma function, with $\Gamma(x+1) = x\,\Gamma(x)$

Q: overall dependence on B? does this make sense?

Q: expected spectral index?

Q: do you expect the signal to be polarized? how?

Source Function

source function

$$S_\nu = \frac{j_\nu}{\alpha_\nu} \propto \frac{\nu^{-(p-1)/2}}{\nu^{-(p+4)/2}} = \nu^{5/2} \tag{42}$$

to see this, recall that

$$j_\nu \;\sim\; \int dE\; N(E)\; P(\nu) \tag{43}$$

$$\alpha_\nu \;\sim\; \nu^{-2} \int dE\; \frac{N(E)}{E}\; P(\nu) \tag{44}$$

thus source function has

$$S_\nu \sim \nu^2 \bar{E} \tag{45}$$

with typical electron energy $\bar{E} = m\bar{\gamma}$ for freq ν
but $\nu(E) \approx \nu_{\mathrm{c}}(E) \sim E^2$, so $\bar{E} \propto \nu^{1/2}$
and thus $S_\nu \sim \nu^{5/2}$ *independent of electron spectral index*

Synchrotron Radiation: the Big Picture

for relativistic electrons with power-law energy distribution

emission coefficient

$$j_\nu \propto \nu^{-(p-1)/2} \tag{46}$$

absorption coefficient

$$\alpha_\nu \propto \nu^{-(p+4)/2} \tag{47}$$

source function (note nonthermal character!)

$$S_\nu \propto \nu^{5/2} \tag{48}$$

Q: optical depth vs ν? implications?
Q: spectrum of a synchrotron emitter?

`www: awesome example: pulsar wind nebulae`
young pulsars are spinning down
much of rotational energy goes into relativistic wind
which collides with the supernova ejecta an emits synchrotron

Director's Cut Extras

Polarization of Synchrotron Radiation

for an electron with a single pitch angle $\tan\alpha = v_\perp / v_\parallel$
$\rightarrow$ circular motion around field line
$\rightarrow$ radiation circularly polarized orthogonal to $\vec{B}$
and elliptically polarized at arbitrary angles

but with distribution of pitch angles α,
elliptical portion cancels out $\rightarrow$ partial **linear polarization**

polarization strength varies with projected angle
of magnetic field on sky
more power orthogonal to projected field direction
$\rightarrow$ net linear polarization, detailed formulae in RL

averaging over power-law distribution of electron energies
partial polarization is $\Pi = (p+1)/(p+7/3)$
and so $\Pi = 3/4$ for $p = 3$: highly polarized!

Transition from Cyclotron to Synchrotron

How and why are the emission spectra so different for cyclotron (non-relativistic) vs synchrotron (relativistic)?

recall: in either case, electron motion is *strictly periodic* with angular frequency

$$\omega_B = \frac{qB \sin \alpha}{mc\gamma} \tag{49}$$

Q: nature of Fourier spectrum of received field?

Q: Fourier spectrum of emission for single pitch angle?

Q: spectrum in nonrelativistic case $\gamma \to 1$?

Q: spectrum in mildly relativistic case?

electron motion at fixed α strictly periodic with ω_B
$\rightarrow$ received field also strictly periodic
$\rightarrow$ Fourier transform of field is nonzero only for
discrete *series* of frequencies $m\omega_B$, $m \in 1, 2, \ldots$

and thus received radiation also is a Fourier series in ω_B

cyclotron = nonrelativistic case: see field $E = E_0 \cos \omega_B t$
Fourier series has *one term*: the fundamental frequency ω_B

when mildly relativistic: Doppler effects add harmonic at $2\omega_B$
and electric field shape modified to sharper, narrower peak

going to strongly relativistic: many harmonics excited
series "envelope" approaches $F(\omega/\omega_c)$
electric field $\rightarrow$ very sharp, very narrow peak

with distribution of pitch angles:
"spaces" in series filled in $\rightarrow$ continuous spectrum

Synchrotron Self-Absorption

Recall strategy so far:

- calculate emission coefficient j_ν
- remember Kirchoff's law $j_\nu = \alpha_\nu \; B_\nu(T)$
- solve for $\alpha_\nu = j_\nu / B_\nu(T)$

We have already found

Q: why won't this work here?

Q: what do we need to do? hint–how did we handle a two-level system?

Kirchoff's law is only good for a *thermal* system
where emitter and absorber particles are nonrelativistic
and have Maxwell-Boltzmann energy/momentum distribution

here: electrons are relativistic and nonthermal

really: Kirchoff is example of *detailed balance*
$\rightarrow$ in equilibrium, emission and absorption rates are the same $\rightarrow$
this still applies in nonthermal case

recall from *2-level* system, with $E_2 = E_1 + h\nu$

$$\alpha_\nu \stackrel{\text{2-level}}{=} \frac{h\nu}{4\pi} \left[n(E_1) B_{12} - n(E_2) B_{21} \right] \; \phi(\nu) \tag{50}$$

Q: physical interpretation of $n(E_1)$? B_{12}? B_{21}? $\phi(\nu)$?

Q: how should this be modified for synchrotron electrons?

in *2-level system*, emission at frequency ν
arises from *unique energy level spacing* $E_2 = E_1 + h\nu$

but cosmic ray electrons have *continuous energy spectrum*
$\rightarrow$ emission at ν can arise from *any two energies*:
generalized to

$$\alpha_\nu = \frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} \left[n(E_1) B_{12} - n(E_2) B_{21} \right] \ \phi_{21}(\nu) \tag{51}$$

- with $\phi_{21}(\nu) \rightarrow \delta[\nu - (E_2 - E_1)/h]$
- first term: true absorption
- second term: stimulated emission

the goal: recast this in terms of what we know
synchrotron emission j_ν

we have

$$\alpha_\nu = \frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} \left[n(E_1) B_{12} - n(E_2) B_{21} \right] \ \phi_{21}(\nu) \tag{52}$$

use Einstein relations, good for thermal and nonthermal

- spontaneous emission rate from state E_2: $A_{21} = 2h\nu^3 B_{21}/c^2$
- absorption and stimulated emission: $B_{21} = B_{12}$

note that spontaneous *emission* is what we know!
we have found synchrotron power $P(\nu, E_2) = 2\pi P(\omega)$,
with E_2 the radiating electron's energy

$$P(\nu, E_2) = h\nu \sum_{E_2} A_{21} \ \phi_{21}(\nu) \tag{53}$$

now impose Einstein conditions and simplify

Q: role of ϕ_{21} and double sum $\sum_{E_1} \sum_{E_2}$?

profile function $\phi_{21}(\nu) \to \delta(E_2 - E_1 - h\nu)$
fixes E_1 for a given E_2 and ν
and double sum $\to$ single sum

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3} \sum_{E_2} [n(E_2 - h\nu) - n(E_2)] \;\; P(\nu, E_2) \tag{54}$$

so far: schematic sum over electron energies
but really a continuum

recall: in each phase space cell h^3
- number of electron states with momentum p is $g_e \; f(p)$
- volume density of states in momentum space volume is d^3p/h^3

and thus

$$\alpha_\nu = g_e \frac{c^2}{8\pi h\nu^3} \frac{1}{h^3} \int [f(p_2^*) - f(p_2)] \;\; P(\nu, E_2) \; d^3p_2 \tag{55}$$

where p_2^* is the momentum corresponding to energy $E_2 - h\nu$

Q: how is f related to electron spectrum $N(E)$?

number of electrons per unit volume
with energy in $(E, E+dE)$ is $N(E)\ dE$

but this means that

$$N(E)\ dE = \frac{4\pi\ g_e}{h^3}\ p^2\ f(p)\ dp \tag{56}$$

and for ultrarelativistic electrons, $E = cp$

thus we have

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3}\int\left[\frac{N(E-h\nu)}{(E-h\nu)^2} - \frac{N(E)}{E^2}\right]\ E^2\ P(\nu, E)\ dE \tag{57}$$

and since $h\nu \ll E$, expand to first order

$$\alpha_\nu = -\frac{c^2}{8\pi\nu^2}\int dE\ P(\nu, E)\ E^2\ \partial_E\left[\frac{N(E)}{E^2}\right] \tag{58}$$

and for a power-law $N(E) \propto E^{-p}$, we have

$$-E^2\partial_E\left[\frac{N(E)}{E^2}\right] = (p+2)\frac{N(E)}{E} \tag{59}$$

Synchrotron Absorption

finally then

$$\alpha_\nu = (p+2)\frac{c^2}{8\pi\nu^2}\int dE\ P(\nu,E)\ \frac{N(E)}{E} \tag{60}$$

note frequency dependence:

- prefactor ν^{-2}
- integral $\int dE\ P(\nu)N(E)/E \sim dE\ P(\nu)E^{-(p+1)} \sim \nu^{-p/2}$

net scaling: $\alpha_\nu \propto \nu^{-(p+4)/2}$

full result

$$\begin{aligned} \alpha_\nu \;=\; & \frac{\sqrt{3}}{8\pi}\Gamma\left(\frac{3p+2}{12}\right)\Gamma\left(\frac{3p+22}{12}\right) \\ & \left(\frac{3q}{2\pi m^3 c^5}\right)^{p/2}\left(\frac{q^3 C}{m}\right)\ (B\sin\alpha)^{(p+2)/2}\ \nu^{-(p+4)/2} \end{aligned}$$