Astronomy 501: Radiative Processes

Lecture 4

Aug 29, 2022

Announcements:

- **Problem Set 1 posted on Canvas, due next Friday**
  you may speak to me, the TA, and other students
- today and Wednesday: **meet in person!**
  please please mask up!

Face-to-Face Introductions

⋆ What do you like to be called?

⋆ Where are you from?

⋆ Research interests (possible or certain)?

⋆ Shareable fun fact?

Last time: resolved vs unresolved sources

Q: what determines which is which?

Q: what's the difference in how they look?

Q: what's the difference in how we quantify them?

Q: what's the difference in what we can learn?

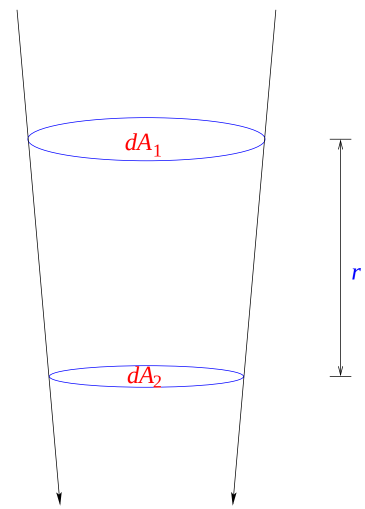
Today: focus on **resolved objects**

Constancy of Specific Intensity in Free Space

in free space: no emission, absorption, scattering

consider rays normal to areas dA_1 and dA_2

separated by a distance r



energy flow is *conserved*, so

$$d\mathcal{E}_1 = I_{\nu_1}\ dA_1\ dt\ d\Omega_1\ d\nu_1 = d\mathcal{E}_2 = I_{\nu_2}\ dA_2\ dt\ d\Omega_2\ d\nu_2$$

- as seen by dA_1, the solid angle $d\Omega_1$ subtended by dA_2 is $d\Omega_1 = dA_2/r^2$, and similarly $d\Omega_2 = dA_1/r^2$ so "etendue" is same: $dA_1 d\Omega_1 = dA_2 d\Omega_2$
- and in free space $d\nu_1 = d\nu_2$, so:

$$I_{\nu_1} = I_{\nu_2} \tag{1}$$

$$I_{\nu_1} = I_{\nu_2} \tag{2}$$

thus: in free space, the intensity is constant along a ray
that is: intensity of an object in free space
is *the same* anywhere along the ray

so along a ray in free space: $I_\nu = \text{constant}$
or along small increment ds of the ray's path

$$\frac{dI_\nu}{ds} \overset{\text{free}}{=} 0 \tag{3}$$

this means: when viewing an object across free space,
the *intensity of the object is constant*
regardless of distance to the object!
$\Rightarrow$ **conservation of surface brightness**

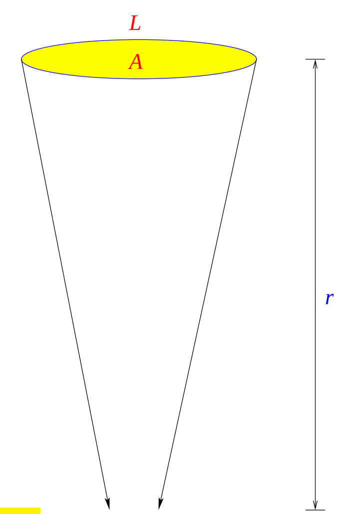
this is huge! and very useful!
Q: what is implied? how can this be true–what about inverse square law? everyday examples?

Conservation of Surface Brightness

consider object in free space at distance r
with luminosity L and projected area $A \perp$ to sightline

flux from source follows usual inverse square

$$F = \frac{L}{4\pi r^2}$$



but *intensity* is flux *per solid angle*
and since $\Omega = A/r^2$, we have

$$I = \frac{F}{\Omega} = \frac{L/4\pi r^2}{A/r^2} = \frac{L}{4\pi A}$$

surface brightness is independent of distance!

and note $I = L/4\pi A$: intensity really is surface brightness
i.e., brightness per unit surface area and solid angle

Consequences of Surface Brightness Conservation

resolved objects in free space
have *same* I at all distances

- Sun's brightness at surface is same as you see in sky
  but at surface subtends 2π steradian – yikes!

- similar planetary nebulae or galaxies all have similar I
  regardless of distance

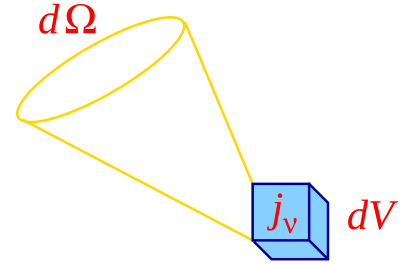
- people and objects across the room don't look $1/r^2$ dimmer
  than those next to you
  fun exercise: when in your everyday life
  do you actually experience the inverse square law for flux?

Adding Sources

matter can act as source and as sink for propagating light

the light energy added by glowing **source** in small volume dV, into a solid angle $d\Omega$, during time interval dt, and in frequency band $(\nu, \nu + d\nu)$, is written

$$d\mathcal{E}_{\rm emit} = j_\nu \; dV \; dt \; d\Omega \; d\nu$$



defines the **emission coefficient**

$$j_\nu = \frac{d\mathcal{E}_{\rm emit}}{dV \; dt \; d\Omega \; d\nu}$$

- power emitted per unit volume, frequency, and solid angle
- cgs units: $[j_\nu] = [\text{erg cm}^{-3} \text{ s}^{-1} \text{ sr}^{-1} \text{ Hz}^{-1}]$
- similarly can define j_λ, and integrated $j = \int j_\nu d\nu$
- *much of the course will be finding j_ν for different situations*

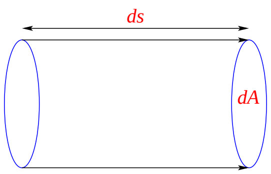
for *isotropic* emitters,
or for distribution of randomly oriented emitters, write

$$j_\nu = \frac{q_\nu}{4\pi} \tag{4}$$

where q_ν is radiated power per unit volume and frequency

sometimes also define *emissivity* $\epsilon_\nu = q_\nu/\rho$
energy emitted per unit freq and mass, with ρ =mass density

beam of area dA going distance ds
has volume $dV = dA\ ds$



so the *energy change* is $d\mathcal{E} = j_\nu\ ds\ dA\ dt\ d\Omega\ d\nu$
and the *intensity change* is

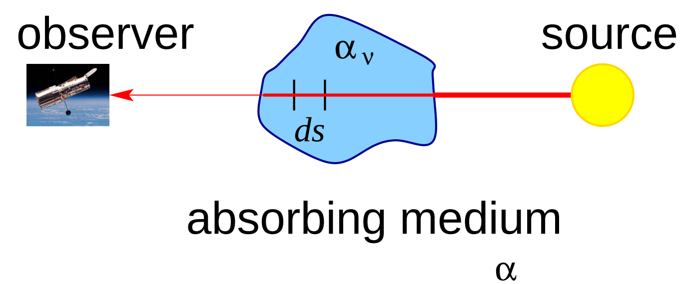
∞

$$dI_\nu \stackrel{\text{sources}}{=} j_\nu\ ds \tag{5}$$

Adding Sinks

as light passes through matter, energy can also be lost due to scattering and/or absorption

we *model* this as follows:



$$dI_\nu = -\alpha_\nu \; I_\nu \; ds$$

features/assumptions:

- losses proportional to distance ds traveled

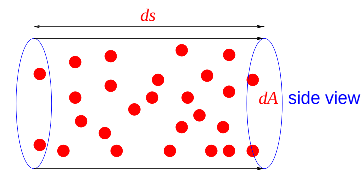
  Q: why is this reasonable?
- losses proportional to intensity

  Q: why is this reasonable?
- defines energy loss per unit pathlength, i.e., **absorption coefficient** α_ν

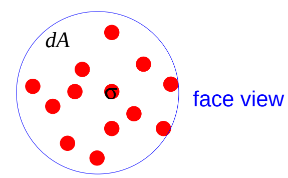
  Q: units/dimensions of α_ν*?*

Absorption Cross Section

consider "absorbers" with a number density n_{a}
each of which presents the beam with an
effective *cross-sectional area* σ_ν



over length ds, number of absorbers is
$dN_{\text{a}} = n_{\text{a}}\, dA\, ds$



a "dartboard problem"– over beam area dA
total "bullseye" area: $\sigma_\nu dN_{\text{a}} = n_{\text{a}}\sigma_\nu\, dA\, ds$

so absorption *probability* is

$$dP_{\text{abs}} = \frac{\text{total bullseye area}}{\text{total beam area}} = n_{\text{a}}\ \sigma_\nu\ ds \qquad (6)$$

Q: for what length ds *does* $P_{\text{abs}} \to 1$*?*
Q: physical significance of $n_{\text{a}}\sigma_\nu$*?*

Cross Sections, Mean Free Path, and Absorption

absorption probability large when photon travels **mean free path**

$$\ell_{\rm mfp} = \frac{1}{n_{\rm a}\sigma_\nu} \tag{7}$$

so we can write $dP_{\rm abs} = ds/\ell_{\rm mfp}$

much of the course will be about σ_ν an its connection to j_ν

and thus beam energy change is

$$d\mathcal{E} = -dP_{\rm abs}\mathcal{E} = -n_{\rm a}\sigma_\nu I_\nu \ ds \ dA \ dt \ d\Omega \ d\nu \tag{8}$$

which must lead to an intensity change

$$dI_\nu \overset{\rm abs}{=} -n_{\rm a} \ \sigma_\nu \ I_\nu \ ds \tag{9}$$

Q: and so?

$$dI_\nu \overset{\text{abs}}{=} -n_{\text{a}} \, \sigma_\nu \, I_\nu \, ds \tag{10}$$

has the expected form, and we identify the absorption coefficient

$$\alpha_\nu = n_{\text{a}} \, \sigma_\nu = \frac{1}{\ell_{\text{mfp}}} \tag{11}$$

note that absorption depends on

- *microphysics* via the cross section σ_ν
- *astrophysics* via density n_{abs} of scatterers

often, write $\alpha_\nu = \rho \kappa_\nu$,
defines **opacity** $\kappa_\nu = (n/\rho)\sigma_\nu \equiv \sigma_\nu/m$
with $m = \rho/n$ the mean mass per absorber

Q: so what determines σ_ν? e.g., for electrons?

Cross Sections

Note that the absorption **cross section** σ_ν is
and *effective* area presented by absorber

for "billiard balls" = neutral, opaque, macroscopic objects
  this is the same as the geometric size
but generally, cross section is *unrelated to geometric size*
  e.g., electrons are point particles (?) but still scatter light

- *generalize* our ideas so that
  $dI_\nu = -n_{\rm a}\ \sigma_\nu\ I_\nu\ ds$ *defines* the cross section
- determined by the details of light-matter interactions
- can be–and usually is!–frequency dependent
- differ according to physical process
  the study of which will be the bulk of this course!

Note: "absorption" here is anything removing energy from beam
  $\rightarrow$ can be true absorption, but also scattering

Putting It All Together

apply **energy conservation** along a pencil of radiation:

$$d\mathcal{E}_{\rm pencil} = -d\mathcal{E}_{\rm absorb} + d\mathcal{E}_{\rm emit} \tag{12}$$

which becomes

$$dI_\nu \; dA \; dt \; d\Omega \; d\nu = -\alpha_\nu \; I_\nu \; ds \; dA \; dt \; d\Omega \; d\nu + j_\nu \; ds \; dA \; dt \; d\Omega \; d\nu$$

and simplifies to

$$dI_\nu = -\alpha_\nu I_\nu \, ds + j_\nu \, ds \tag{13}$$

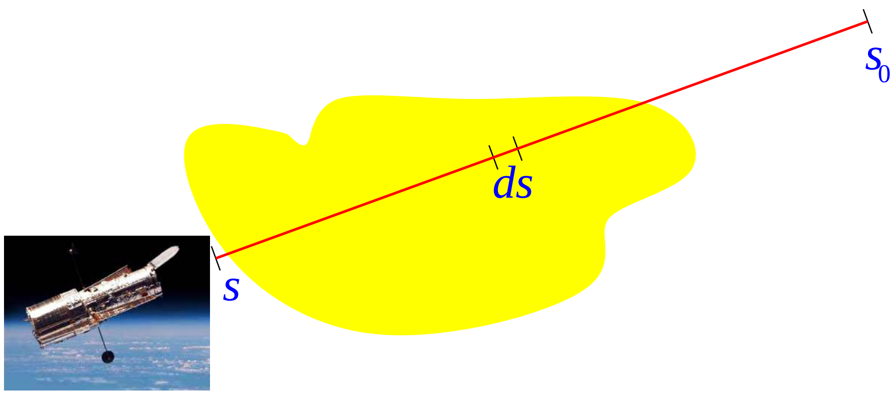
this is a Big Deal!

Q: why?

The Equation of Radiative Transfer

the mighty **equation of radiative transfer**

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu \tag{14}$$



- fundamental equation in this course
- physical meaning: things look (I_ν) the way they do due to sources and along each sightline
- *sources* parameterized via j_ν
- *sinks* parameterized via $\alpha_\nu = n_\mathrm{a}\, \sigma_\nu = \rho \kappa_\nu = 1/\ell_{\mathrm{mfp},\nu}$

Transfer Equation: Limiting Cases

equation of radiative transfer:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu \tag{15}$$

Sources but no Sinks

if sources exist but there are no sinks: $\alpha_\nu = 0$

$$\frac{dI_\nu}{ds} = j_\nu \tag{16}$$

solve along path starting at sightline distance s_0:

$$I_\nu(s) = I_\nu(s_0) + \int_{s_0}^{s} j_\nu \; ds' \tag{17}$$

- the *increment* in intensity is due to integral of sources *along sightline*
- for $j_\nu \to 0$: free space case and $I_\nu(s) = I_\nu(s_0)$: recover surface brightness conservation!

Special Case: Sinks but no Sources

if absorption only, no sources: $j_\nu = 0$

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu \tag{18}$$

and so on a sightline from s_0 to s

$$I_\nu(s) = I_\nu(s_0)\ e^{-\int_{s_0}^{s} \alpha_\nu\ ds'} \tag{19}$$

- intensity *decrement* is *exponential*!
- exponent depends on line integral of absorption coefficient

useful to define **optical depth** via $d\tau_\nu \equiv \alpha_\nu\ ds$

$$\tau_\nu(s) = \int_{s_0}^{s} \alpha_\nu\ ds' = \int_{s_0}^{s} \frac{ds'}{\ell_{\mathrm{mfp},\nu}} \tag{20}$$

and thus *for absorption only* $I_\nu(s) = I_\nu(s_0) e^{-\tau_\nu(s)}$

Optical Depth

optical depth, in terms of cross section

$$\tau_\nu(s) = \int_{s_0}^{s} n_{\rm a}\, \sigma_\nu \, ds' = \int_{s_0}^{s} \frac{ds'}{\ell_{{\rm mfp},\nu}} \quad (21)$$

$$= \text{number of mean free paths} \quad (22)$$

optical depth counts mean free paths along sightline
i.e., typical number of absorption events

Limiting cases:

• $\tau_\nu \ll 1$: **optically thin**
absorption unlikely $\rightarrow$ **transparent**

• $\tau_\nu \gg 1$: **optically thick**
absorption overwhelmingly likely $\rightarrow$ **opaque**

`www: Pillars of creation: Optical and IR`

Q: what features are optically thick?

Q: what features are optically thin?

Column Density

Note "separation of variables" in optical depth

$$\tau_\nu(s) \;=\; \underbrace{\sigma_\nu}_{\text{microphysics}} \;\; \underbrace{\int_{s_0}^{s} n_{\mathrm{a}}(s')\ ds'}_{\text{astrophysics}} \tag{23}$$

From observations, can (sometimes) infer τ_ν *Q: how?*
but cross section σ_ν fixed by absorption microphysics
i.e., by theory and/or lab data

absorber astrophysics controlled by **column density**

$$N_a(s) \equiv \int_{s_0}^{s} n_{\mathrm{a}}(s')\ ds' \tag{24}$$

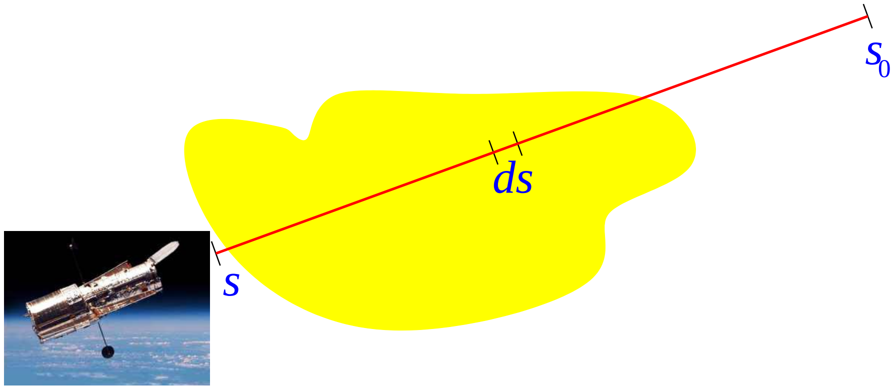
line integral of number density over entire line of sight s
cgs units $[N_{\mathrm{a}}] = [\mathrm{cm}^{-2}]$

Q: what does column density represent physically?

column density

$$N_a(s) \equiv \int_{s_0}^{s} n_{\rm a} \; ds'$$

so $\tau_\nu = \sigma_\nu N_{\rm a}$



- column density is projection of 3-D absorber density onto 2-D sky, "collapsing" the sightline "cosmic roadkill"

- if source is a slab $\perp$ to sightline, then $N_{\rm a}$ is *absorber surface density*

- if source is multiple slabs $\perp$ to sightline, then $N_{\rm a}$ sums surface density of all slabs

Q: from $N_{\rm a}$, how to recover 3-D density n_a?

Radiation Transfer Equation, Formal Solution

equation of transfer

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu \tag{25}$$

divide by α_ν and rewrite
in terms of optical depth $d\tau_\nu = \alpha_\nu ds$

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu \tag{26}$$

with the **source function**

$$\boxed{S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{j_\nu}{n_\mathrm{a}\sigma_\nu}} \tag{27}$$

Q: source function dimensions?

Source Function

$S_\nu = j_\nu/\alpha_\nu$ has dimensions of surface brightness
What does it represent physically?

consider the case where the *same* matter
is responsible for both emission and absorption; then:

- $\alpha_\nu = n\sigma_\nu$, with n the particle number density
- $j_\nu = n\, dL_\nu/d\Omega$, with $dL_\nu/d\Omega$ the specific power emitted *per particle* and per solid angle

and thus we have

$$S_\nu = \frac{dL_\nu/d\Omega}{\sigma_\nu} \tag{28}$$

specific power per unit effective area and solid angle
$\rightarrow$ effective surface brightness of each particle!

spoiler alert: S_ν encodes emission vs absorption relation
ultimately set by quantum mechanical symmetries
e.g., time reversal invariance, "detailed balance"

Radiative Transfer Equation: Formal Solution

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu \tag{29}$$

If emission *independent* of I_ν (*not* always true! *Q: why?*)
Then can formally solve

Write $I_\nu = \Phi_\nu e^{-\tau_\nu}$, i.e., use *integrating factor* $e^{-\tau_\nu}$, so

$$\frac{d(\Phi_\nu e^{-\tau_\nu})}{d\tau_\nu} = e^{-\tau_\nu}\frac{d\Phi_\nu}{d\tau_\nu} - \Phi_\nu e^{-\tau_\nu} \tag{30}$$

$$= -\Phi_\nu e^{-\tau_\nu} + S_\nu \tag{31}$$

and so we have

$$\frac{d\Phi_\nu}{d\tau_\nu} = e^{+\tau_\nu} S_\nu(\tau_\nu) \tag{32}$$

and thus

$$\Phi_\nu(s) = \Phi_\nu(0) + \int_0^{\tau_\nu} e^{\tau'_\nu}\ S_\nu(\tau'_\nu)\ d\tau'_\nu \tag{33}$$

$$\Phi_\nu(s) = \Phi_\nu(0) + \int_0^{\tau_\nu(s)} e^{\tau'_\nu}\ S_\nu(\tau'_\nu)\ d\tau'_\nu \tag{34}$$

and then

$$\begin{aligned} I_\nu(s) &= \Phi_\nu(s)\ e^{-\tau_\nu(s)} &(35)\\ &= I_\nu(0)\ e^{-\tau_\nu(s)} + \int_0^{\tau_\nu(s)} e^{-[\tau_\nu(s)-\tau'_\nu]}\ S_\nu(\tau'_\nu)\ d\tau'_\nu &(36)\end{aligned}$$

in terms of original variables

$$I_\nu(s) = I_\nu(0) e^{-\tau_\nu(s)} + \int_{s_0}^{s} e^{-[\tau_\nu(s)-\tau_\nu(s')]}\ j_\nu(\tau'_\nu)\ ds'$$

Q: what strikes you about these solutions?

Formal solution to transfer equation:

$$I_\nu(s) = I_\nu(0)\ e^{-\tau_\nu(s)} + \int_0^{\tau_\nu(s)} e^{-[\tau_\nu(s)-\tau'_\nu]}\ S_\nu(\tau'_\nu)\ d\tau'_\nu \qquad (37)$$

in terms of original variables

$$I_\nu(s) = I_\nu(0)e^{-\tau_\nu(s)} + \int_{s_0}^{s} e^{-[\tau_\nu(s)-\tau_\nu(s')]}\ j_\nu(s')\ ds'$$

- *first term:*
  initial intensity degraded by absorption

- *second term:*
  added intensity depends on sources along column
  but optical depth weights against sources with $\tau_\nu \gtrsim 1$

Formal Solution: Special Cases

For spatially *constant* source function $S_\nu = j_\nu/\alpha_\nu$:

$$I_\nu(s) = e^{-\tau_\nu(s)} I_\nu(0) + S_\nu \int_0^{\tau_\nu(s)} e^{-[\tau_\nu(s)-\tau'_\nu]} \, d\tau'_\nu \qquad (38)$$

$$= e^{-\tau_\nu(s)} I_\nu(0) + \left(1 - e^{-\tau_\nu(s)}\right) S_\nu \qquad (39)$$

- *optically thin:* $\tau_\nu \ll 1$
  $I_\nu \approx (1-\tau_\nu) I_\nu(0) + \tau_\nu S_\nu$
- *optically thick:* $\tau_\nu \gg 1$
  $I_\nu \to S_\nu$
  $\Rightarrow$ optically thick intensity is source function!

what's going on? rewrite:

$$\frac{dI_\nu}{ds} = -\frac{1}{\ell_{\mathrm{mfp},\nu}} (I_\nu - S_\nu) \qquad (40)$$

Q: what happens if $I_\nu < S_\nu$? if $I_\nu > S_\nu$?
Q: lesson? characteristic scales?

Radiation Transfer as Relaxation

$$\frac{dI_\nu}{ds} = -\frac{1}{\ell_{\mathrm{mfp},\nu}}(I_\nu - S_\nu) \tag{41}$$

- if $I_\nu < S_\nu$, then $dI_\nu/ds > 0$:
  $\rightarrow$ intensity *increases* along path
- if $I_\nu > S_\nu$, intensity *decreases*

equation is "*self regulating*!"
I_ν "*relaxes*" to "attractor" S_ν

and characteristic lengthscale for relaxation is mean free path!
recall $S_\nu = \ell_{\mathrm{mfp},\nu} j_\nu$: this is *"source-only"* result
for sightline pathlength $s = \ell_{\mathrm{mfp},\nu}$

Director's Cut Extras

Optical Depth and Mean Free Path

Average optical depth is

$$\langle \tau_\nu \rangle = \frac{\int \tau_\nu e^{-\tau_\nu} d\tau_\nu}{\int e^{-\tau_\nu} d\tau_\nu} = 1$$

for constant density $n_{\rm a}$, this occurs
at the **mean free path**

$$\ell_{\rm mfp,\nu} = \frac{1}{n_{\rm a}\ \sigma_\nu}$$

average distance between collisions

similarly *mean free time* between collisions

$$\tau_\nu = \frac{\ell_{\rm mfp,\nu}}{c} \tag{42}$$

where we used $v = c$ for all photons

Radiative Forces

generalize our definition of flux:
energy flux in direction $\hat{n}$ is

$$\vec{F}_\nu = \int I_\nu \; \hat{n} \; d\Omega \tag{43}$$

recovers old result if we take $\hat{z} \cdot \vec{F}_\nu$

each photon has momentum E/c, and so
momentum per unit area and pathlength
absorbed by medium with absorption coefficient α_ν:

$$\vec{\mathcal{F}} = \frac{d\vec{p}}{dt \; dA \; ds} = \frac{1}{c} \int \alpha_\nu \; \vec{F}_\nu \; d\nu \tag{44}$$

but $dA \; ds = dV$, and $d\vec{p}/dt$ is force,
so $\vec{\mathcal{F}}$ is the **force density**
i.e., force per unit volume, on absorbing matter

force per unit mass is

$$\vec{f} = \frac{\vec{\mathcal{F}}}{\rho} = \frac{1}{c} \int \kappa_\nu \ \vec{F}_\nu \ d\nu \tag{45}$$

Note: we have accounted only force due to *absorption* of radiation

What about *emission?*

If emission is isotropic, no net force
if not, must include this as a separate term