Astronomy 501: Radiative Processes

Lecture 5

Aug 31, 2022

Announcements:

- **Problem Set 1 posted on Canvas, due Friday 5pm**
- Office hours: instructor–after class today, or by appointment
  TA: Tomorrow 11:30-12:30
- today: **meet in person!**
  please please mask up!

Last time: thanks for the great questions and discussion!
ingredients of radiative transfer

- free space *Q: meaning?* I_ν *result? significance?*
- emission *Q: how quantified? physical origin?*
- absorption *Q: how quantified? physical origin?*

the mighty equation of radiation transfer
Q: what is it?

Special Case: Sinks but no Sources

if absorption only, no sources: $j_\nu = 0$

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu \tag{1}$$

and so on a sightline from s_0 to s

$$I_\nu(s) = I_\nu(s_0)\ e^{-\int_{s_0}^{s} \alpha_\nu\ ds'} \tag{2}$$

- intensity *decrement* is *exponential*!
- exponent depends on line integral of absorption coefficient

useful to define **optical depth** via $d\tau_\nu \equiv \alpha_\nu\ ds$

$$\tau_\nu(s) = \int_{s_0}^{s} \alpha_\nu\ ds' = \int_{s_0}^{s} \frac{ds'}{\ell_{\mathrm{mfp},\nu}} \tag{3}$$

and thus *for absorption only* $I_\nu(s) = I_\nu(s_0) e^{-\tau_\nu(s)}$

Optical Depth

optical depth, in terms of cross section

$$\tau_\nu(s) = \int_{s_0}^{s} n_{\text{a}}\, \sigma_\nu\, ds' = \int_{s_0}^{s} \frac{ds'}{\ell_{\text{mfp},\nu}} \tag{4}$$

$$= \text{number of mean free paths} \tag{5}$$

optical depth counts mean free paths along sightline
i.e., typical number of absorption events

Limiting cases:

- $\tau_\nu \ll 1$: **optically thin**
  absorption unlikely $\to$ **transparent**

- $\tau_\nu \gg 1$: **optically thick**
  absorption overwhelmingly likely $\to$ **opaque**

Column Density

Note "separation of variables" in optical depth

$$\tau_\nu(s) \ = \ \underbrace{\sigma_\nu}_{\text{microphysics}} \ \underbrace{\int_{s_0}^{s} n_{\mathrm{a}}(s') \ ds'}_{\text{astrophysics}} \tag{6}$$

From observations, can (sometimes) infer τ_ν *Q: how?*
but cross section σ_ν fixed by absorption microphysics
i.e., by theory and/or lab data

absorber astrophysics controlled by **column density**

$$N_a(s) \equiv \int_{s_0}^{s} n_{\mathrm{a}}(s') \ ds' \tag{7}$$

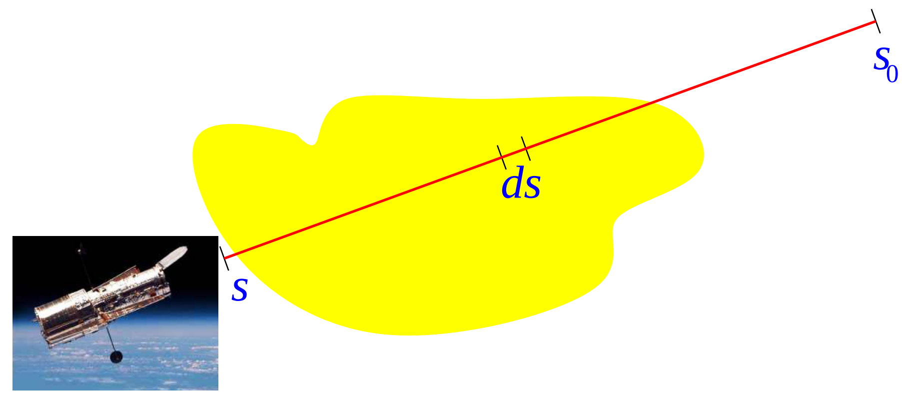
line integral of number density over entire line of sight s
cgs units $[N_{\mathrm{a}}] = [\mathrm{cm}^{-2}]$

Q: what does column density represent physically?

column density

$$N_a(s) \equiv \int_{s_0}^{s} n_{\mathrm{a}} \; ds'$$

so $\tau_\nu = \sigma_\nu N_{\mathrm{a}}$



- column density is projection of 3-D absorber density onto 2-D sky, “collapsing” the sightline “cosmic roadkill”

- if source is a slab $\perp$ to sightline, then N_{a} is *absorber surface density*

- if source is multiple slabs $\perp$ to sightline, then N_{a} sums surface density of all slabs

Q: from N_{a}, how to recover 3-D density n_a?

Radiation Transfer Equation, Formal Solution

equation of transfer

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu \tag{8}$$

divide by α_ν and rewrite

in terms of optical depth $d\tau_\nu = \alpha_\nu ds$

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu \tag{9}$$

with the **source function**

$$\boxed{S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{j_\nu}{n_{\mathrm{a}}\sigma_\nu}} \tag{10}$$

Q: source function dimensions?

Source Function

$S_\nu = j_\nu/\alpha_\nu$ has dimensions of surface brightness
What does it represent physically?

consider the case where the *same* matter
is responsible for both emission and absorption; then:

- $\alpha_\nu = n\sigma_\nu$, with n the particle number density
- $j_\nu = n\, dL_\nu/d\Omega$, with $dL_\nu/d\Omega$ the specific power emitted *per particle* and per solid angle

and thus we have

$$S_\nu = \frac{dL_\nu/d\Omega}{\sigma_\nu} \tag{11}$$

specific power per unit effective area and solid angle
$\rightarrow$ effective surface brightness of each particle!

spoiler alert: S_ν encodes emission vs absorption relation
ultimately set by quantum mechanical symmetries
e.g., time reversal invariance, "detailed balance"

Radiative Transfer Equation: Formal Solution

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu \tag{12}$$

If emission *independent* of I_ν (*not* always true! *Q: why?*)
Then can formally solve

Write $I_\nu = \Phi_\nu e^{-\tau_\nu}$, i.e., use *integrating factor* $e^{-\tau_\nu}$, so

$$\frac{dI_\nu}{ds} = \frac{d(\Phi_\nu e^{-\tau_\nu})}{d\tau_\nu} = e^{-\tau_\nu}\frac{d\Phi_\nu}{d\tau_\nu} - \Phi_\nu e^{-\tau_\nu} \tag{13}$$

$$= -I_\nu + S_\nu = -\Phi_\nu e^{-\tau_\nu} + S_\nu \tag{14}$$

and so we have

$$\frac{d\Phi_\nu}{d\tau_\nu} = e^{+\tau_\nu} S_\nu(\tau_\nu) \tag{15}$$

 and thus

$$\Phi_\nu(s) = \Phi_\nu(0) + \int_0^{\tau_\nu} e^{\tau'_\nu}\ S_\nu(\tau'_\nu)\ d\tau'_\nu \tag{16}$$

$$\Phi_\nu(s) = \Phi_\nu(0) + \int_0^{\tau_\nu(s)} e^{\tau'_\nu} \; S_\nu(\tau'_\nu) \; d\tau'_\nu \tag{17}$$

and then

$$I_\nu(s) \;=\; \Phi_\nu(s) \; e^{-\tau_\nu(s)} \tag{18}$$

$$\;=\; I_\nu(0) \; e^{-\tau_\nu(s)} + \int_0^{\tau_\nu(s)} e^{-[\tau_\nu(s)-\tau'_\nu]} \; S_\nu(\tau'_\nu) \; d\tau'_\nu \tag{19}$$

in terms of original variables

$$I_\nu(s) = I_\nu(0) e^{-\tau_\nu(s)} + \int_{s_0}^{s} e^{-[\tau_\nu(s)-\tau_\nu(s')]} \; j_\nu(\tau'_\nu) \; ds'$$

Q: what strikes you about these solutions?

Formal solution to transfer equation:

$$I_\nu(s) = I_\nu(0)\ e^{-\tau_\nu(s)} + \int_0^{\tau_\nu(s)} e^{-[\tau_\nu(s)-\tau_\nu']}\ S_\nu(\tau_\nu')\ d\tau_\nu' \qquad (20)$$

in terms of original variables

$$I_\nu(s) = I_\nu(0) e^{-\tau_\nu(s)} + \int_{s_0}^{s} e^{-[\tau_\nu(s)-\tau_\nu(s')]}\ j_\nu(s')\ ds'$$

- *first term:*
  initial ("background") intensity degraded by absorption

- *second term:*
  sources along column add intensity
  but optical depth suppresses sources with $\tau_\nu \gtrsim 1$

Formal Solution: Special Cases

For spatially *constant* source function $S_\nu = j_\nu/\alpha_\nu$:

$$I_\nu(s) = e^{-\tau_\nu(s)} I_\nu(0) + S_\nu \int_0^{\tau_\nu(s)} e^{-[\tau_\nu(s)-\tau'_\nu]}\, d\tau'_\nu \quad (21)$$

$$= e^{-\tau_\nu(s)} I_\nu(0) + \left(1 - e^{-\tau_\nu(s)}\right) S_\nu \quad (22)$$

- *optically thin:* $\tau_\nu \ll 1$
  $I_\nu \approx (1-\tau_\nu) I_\nu(0) + \tau_\nu S_\nu$
- *optically thick:* $\tau_\nu \gg 1$
  $I_\nu \to S_\nu$
  $\Rightarrow$ optically thick intensity is source function!

what's going on? rewrite:

$$\frac{dI_\nu}{ds} = -\frac{1}{\ell_{\mathrm{mfp},\nu}} (I_\nu - S_\nu) \quad (23)$$

Q: what happens if $I_\nu < S_\nu$? if $I_\nu > S_\nu$?
Q: lesson? characteristic scales?

Radiation Transfer as Relaxation

$$\frac{dI_\nu}{ds} = -\frac{1}{\ell_{\mathrm{mfp},\nu}} (I_\nu - S_\nu) \tag{24}$$

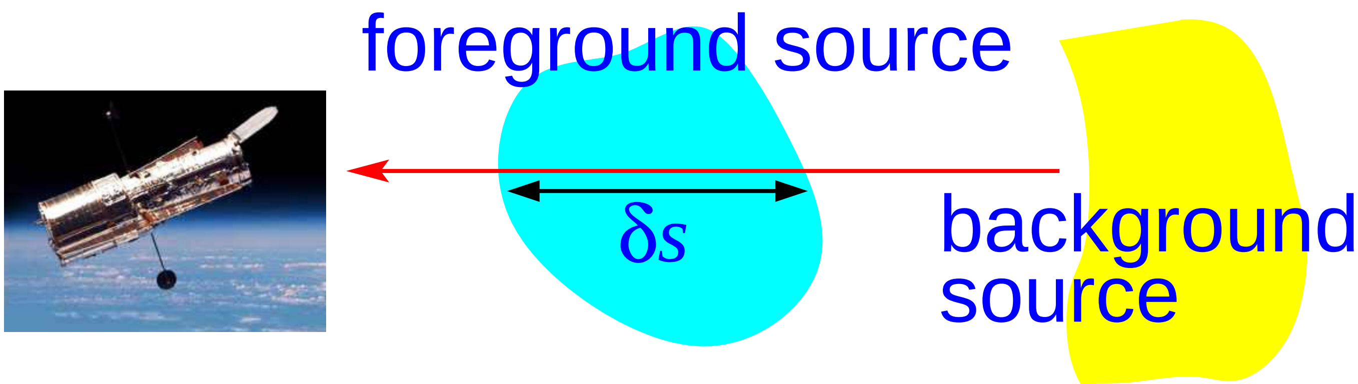
- if $I_\nu < S_\nu$, then $dI_\nu/ds > 0$:
  $\rightarrow$ intensity *increases* along path
- if $I_\nu > S_\nu$, intensity *decreases*

equation is "*self regulating*!"
I_ν "*relaxes*" to "attractor" S_ν

and characteristic lengthscale for relaxation is mean free path!
recall $S_\nu = \ell_{\mathrm{mfp},\nu} \, j_\nu$: this is *"source-only"* result
for sightline pathlength $s = \ell_{\mathrm{mfp},\nu}$

Now imagine:

- *optically thin* foreground source
- and (possibly) a background source

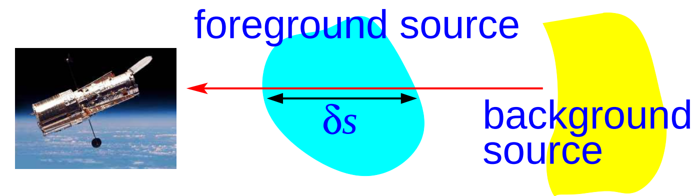


Q: What do we see if no background? if there is one?

Q: Everyday examples?

An Optically Thin Source

- $\tau_\nu \ll 1$: **optically thin** = **transparent**
- consider thin foreground source j_ν illuminated by background $I_\nu(0)$



$$I_\nu \approx (1 - \tau_\nu)\, I_\nu(0) + j_\nu\, \delta s$$

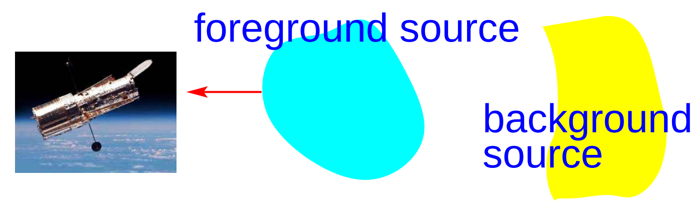
- **physical interpretation:** observed intensity combines slightly diminished background emission + foreground source along sightline
- **sky view**: foreground object with background shining through
- *all* of foreground source volume is projected on sky! useful for probing source interior and global properties

Everyday examples: air, thin smoke ... but these are really scattering

Q: compare/contrast case of optically thick foreground source: physical meaning? what see? what learn?

An Optical Thick Source

- $\tau_\nu \gg 1$: **optically thick** = **opaque**



$$I_\nu \to S_\nu = j_\nu \, \ell_{\rm mfp,\nu}$$

optically thick intensity is source function!

- **sky view**: source **surface**, if solid
  or outermost skin to depth $\ell_{\rm mfp,\nu}$
- measure surface S_ν
- *no information* about interior or background

Everyday examples: most solid objects, deep/muddy water

Note: α_ν and thus τ_ν spectral dependence can mean
the same object can be thin at some ν and thick at others!

Optical Depth and Astrophysical Objects

Q: examples of resolved optically thick astronomical objects?

Q: examples of resolved optically thin *astronomical objects?*

Q: Observe and interpret:

`www: supernova remnants in optical`

`www: Orion nebula in optical`

`www: multiwavelength dark cloud Barnard 68`

`www: galaxies`

`www: all sky: optical, microwave, near infrared`

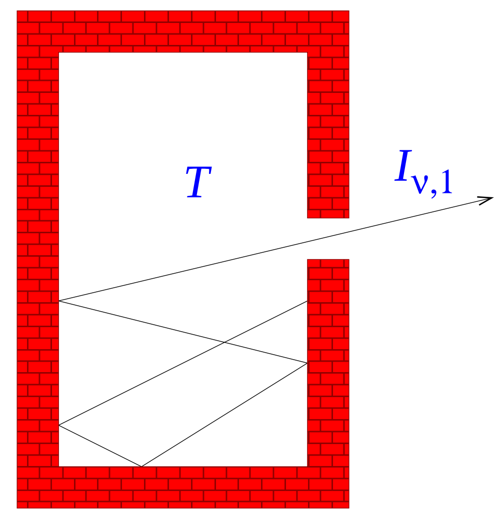
Blackbody Radiation

Radiation and Thermodynamics

consider an enclosure ("*box 1*")
in *thermodynamic equilibrium* at temperature T

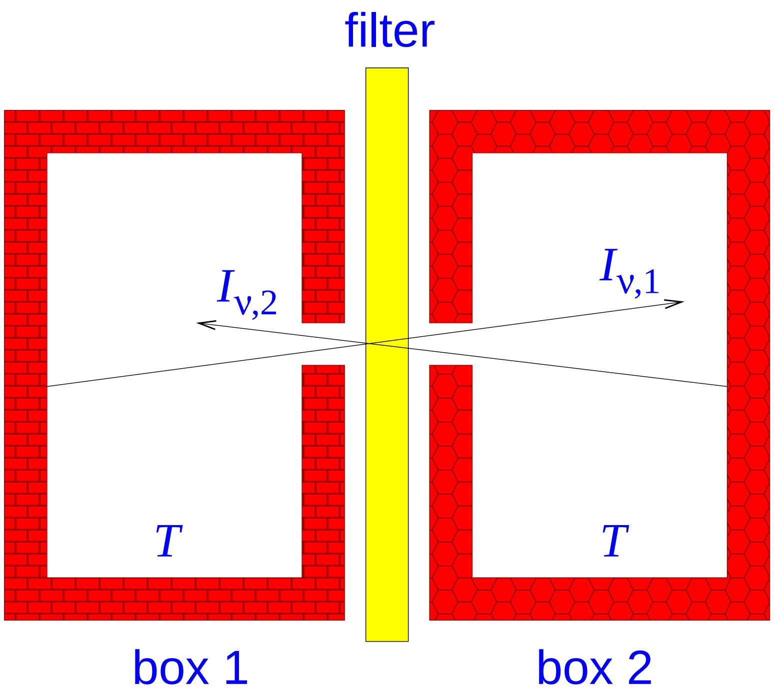
the matter in box 1
- is in random thermal motion
- will absorb and emit radiation
  details of which depends on
  the details of box material and geometry
- but equilibrium
  $\rightarrow$ radiation field in box doesn't change

open little hole: escaping radiation has intensity $I_{\nu,1}$



box 1

now add another enclosure ("box 2"), also at temperature T
but made of *different material*



separate boxes by *filter passing only frequency* ν
radiation from each box incident on screen
Q: imagine $I_{\nu,1} > I_{\nu,2}$*; what happens?*
Q: lesson?

Blackbody Radiation

if both boxes at *same T* $\Rightarrow$ *no net energy transfer*

but this requires $I_{\nu,1} = I_{\nu,2}$ and so the radiation is:

- independent of the composition of the box
- a universal function of T
- **blackbody radiation** with intensity $I_\nu^{\rm blackbody} \equiv B_\nu(T)$

Spoiler alert (useful for PS1): blackbody radiation

$$B_\nu(T) = \frac{2h}{c^2} \frac{\nu^3}{e^{h\nu/kT} - 1} \tag{25}$$

with $h =$ Planck's constant, $k =$ Boltzmann's constant

in wavelength space

$$B_\lambda(T) = 2hc^2 \frac{\lambda^{-5}}{e^{hc/\lambda kT} - 1} \tag{26}$$

blackbody integrated intensity:

$$B(T) = \int B_\nu(T)\ d\nu = \int B_\lambda(T)\ d\lambda \tag{27}$$

$$= \frac{2\pi^4}{15}\frac{k^4T^4}{c^3h^3} = \frac{\sigma_{\rm SB}}{\pi}T^4 = \frac{c}{4\pi}aT^4 \tag{28}$$

blackbody flux

$$F_\nu(T) = \pi B_\nu(T) = \frac{2\pi h}{c^2}\frac{\nu^3}{e^{h\nu/kT} - 1} \tag{29}$$

$$F(T) = \pi B(T) \equiv \sigma_{\rm SB}T^4 = \frac{2\pi^5}{15}\frac{k^4T^4}{c^2h^3} \tag{30}$$

defines *Stefan-Boltzmann constant*

$$\sigma_{\rm SB} = \frac{2\pi^5}{15}\frac{k^4}{c^2h^3} = 5.670 \times 10^{-5}\ \text{erg cm}^{-2}\ \text{s}^{-1}\ \text{K}^{-4} \tag{31}$$

Q: to order of magnitude: integrated number density?

Note: *blackbody quantities determined entirely by T*
no adjustable parameters!

mean number density: dimensions $[n] = [\text{length}^{-3}]$
can only depend on T, and physical constants h, c, k
can form only one length: $[hc/kT] = [\text{length}]$
$\rightarrow$ expect $n \sim (hc/kT)^3$

photon number density

$$n_\nu(T) = \frac{4\pi B_\nu(T)}{hc\nu} = \frac{8\pi}{c^3}\frac{\nu^2}{e^{h\nu/kT} - 1} \quad (32)$$

$$n(T) = 16\pi\zeta(3)\left(\frac{kT}{hc}\right)^3 \quad (33)$$

where $\zeta(3) = 1 + 1/2^3 + 1/3^3 + 1/4^3 + \cdots = 1.2020569\ldots$

Q: implications–what does and doesn't n depend on?

blackbody photon number density

$$n(T) = 16\pi\zeta(3)\left(\frac{kT}{hc}\right)^3 \quad (34)$$

i.e., $n \propto T^3$

So if temperatures changes, photon number changes

blackbody photon number is not conserved

photons massless $\rightarrow$ can always make more!

if heat up, photon number increases

and spectrum relaxes to blackbody form

blackbody energy density?

to order of magnitude, expect $u \sim nkT \sim (kT)^4/(hc)^3$

integrated energy density

$$u_\nu(T) = \frac{4\pi B_\nu(T)}{c} = \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/kT} - 1} \tag{35}$$

$$u(T) = \frac{4\pi B(T)}{c} = \frac{8\pi^5}{15} \frac{k^4 T^4}{c^3 h^3} \tag{36}$$

$$\equiv aT^4 = \frac{4\sigma_{\rm SB}}{c} T^4 \tag{37}$$

defines *Stefan-Boltzmann radiation density constant* $a = 4\sigma_{\rm SB}/c$

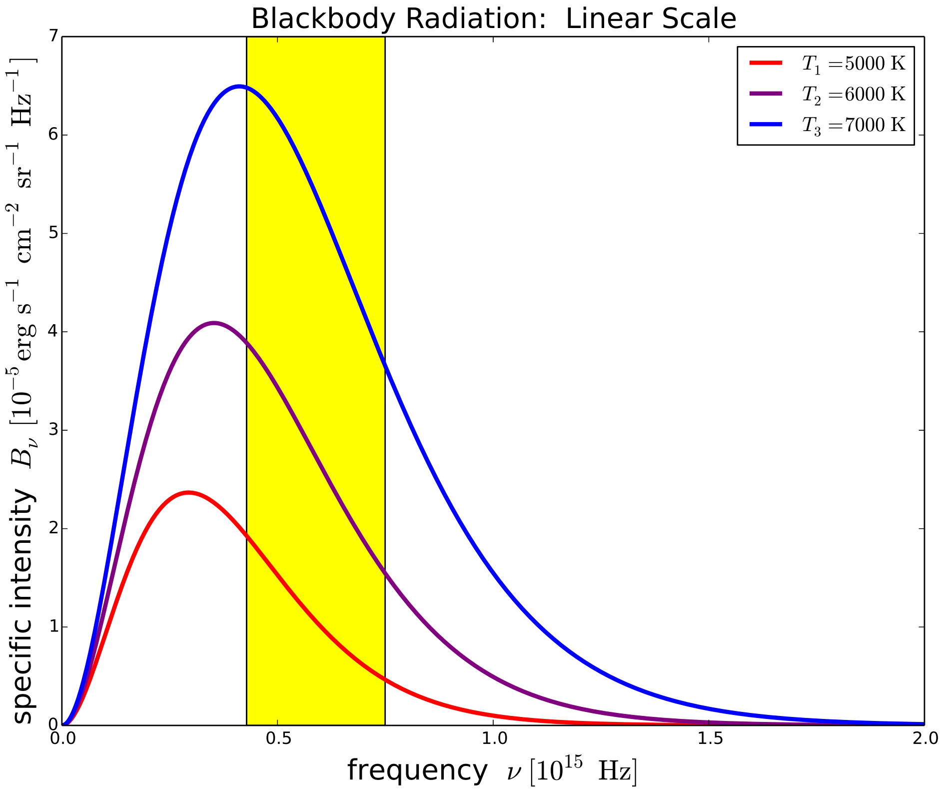
mean photon energy:
only one way to form an energy
$\rightarrow$ expect $\langle E \rangle \sim kT$

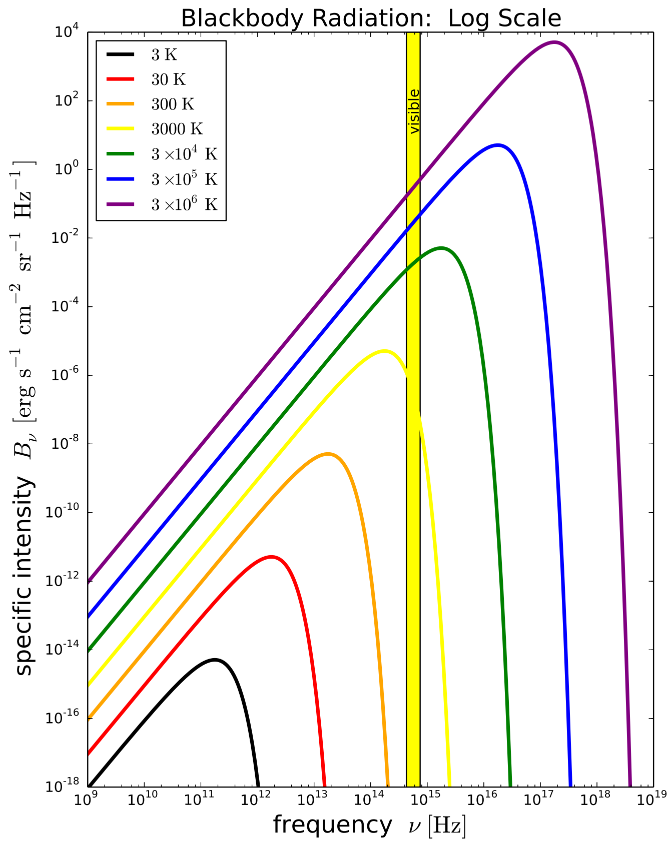
exact result:

$$\langle E \rangle \equiv \frac{u(T)}{n(T)} \tag{38}$$

$$= \frac{\pi^4}{30\zeta(3)} kT = 2.701\, kT \tag{39}$$

Blackbody Spectral Properties





plots of B_ν vs ν *Q: what strikes you?*

Wien's Displacement Laws

for blackbodies,
specific intensity, and flux, and energy density have

$$I_\nu \propto F_\nu \propto u_\nu \propto \frac{\nu^3}{e^{h\nu/kT} - 1} \quad (40)$$

at fixed T, these *spectra all peak at same frequency*

maximum when $x = h\nu/kT$ satisfies $x = 3(1 - e^{-x})$
$\rightarrow x_{\rm max} = 2.821439\ldots$, which gives

$$\frac{\nu_{\rm max}}{T} = x_{\rm max}\frac{k}{h} = 5.88 \times 10^{10} \text{ Hz K}^{-1} \quad (41)$$

i.e., $\nu_{\rm max} \propto T$, as expected from dimensional analysis in wavelength space, $I_\lambda \propto \lambda^{-5}/(e^{hc/\lambda kT} - 1)$

maximum when $y = hc/\lambda kT$ satisfies $y = 5(1 - e^{-y})$
$\rightarrow y_{\rm max} = 4.9651\ldots$, which gives

$$\lambda_{\rm max}\, T = \frac{1}{y_{\rm max}} \frac{hc}{k} = 0.290 \text{ cm K} \qquad (42)$$

i.e., $\lambda_{\rm max} \propto 1/T$, as expected from dimensional analysis

both versions of Wien's Law measure T: **color temperature**

crucial gotcha: **beware!** $\lambda_{\rm max} \neq c/\nu_{\rm max}$
Q: why?