Astronomy 501: Radiative Processes

Lecture 6

Sept 2, 2022

Announcements:

- **Problem Set 1 due today on Canvas at 5pm**
- **Problem Set 2 out today, due next Friday Sept 9**
- Thanks to all for making the most of face-to-face! Please keep up the great discussion!
- Planet imaging is not just for homework
  `www: first JWST detection`

Last time:

Optical thickness and imaging

Q: what defines optical thickness? thin/thick distinction?

Q: what do you see when looking at an optically thick source?

Began blackbody radiation:

Q: Why is B_ν universal? what does B_ν depend on?

optical depth

$$\tau_\nu(s) = \underbrace{\sigma_\nu}_{\text{microphysics}} \underbrace{\int_{s_0}^{s} n_a(s')\, ds'}_{\text{astrophysics}} \tag{1}$$

Limiting cases:

• $\tau_\nu \ll 1$: **optically thin**
absorption unlikely $\rightarrow$ **transparent**
see entire source volume, projected on sky

• $\tau_\nu \gg 1$: **optically thick**
absorption overwhelmingly likely $\rightarrow$ **opaque**
see source function $S_\nu = j_\nu/\alpha_\nu$ of last few mean free paths

absorber astrophysics controlled by **column density**

$$N_a(s) \equiv \int_{s_0}^{s} n_a(s')\, ds' \tag{2}$$

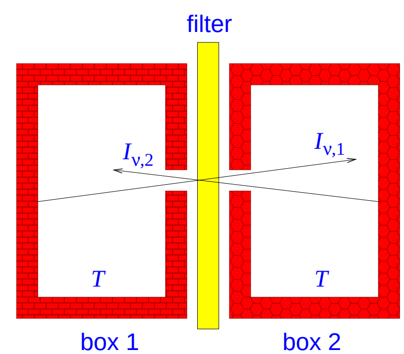
line integral of number density over entire line of sight s

Blackbody Radiation

useful to define idealized substance: **blackbody**

- object in thermodynamic equilibrium at temperature T
- absorbs all radiation at all ν, emits according to T
- why radiate? constituent particles in thermal motion collisions/interactions lead to emission (more on this soon!)

two boxes at *same* $T \Rightarrow$ *no net energy transfer*
but this requires $I_{\nu,1} = I_{\nu,2}$ and so the radiation is:



- independent of the composition of the box
- a universal function of T
- *isotropic Q: why?*
- **blackbody radiation** with intensity $I_\nu^{\text{blackbody}} \equiv B_\nu(T)$

Radiation Thermodynamics

experiment and everyday life show: *hot objects glow*

- *radiation participates in energy exchange, is a form of heat*
- *we must include radiation in thermodynamics!*

We have seen: radiation has both an *energy density* u
and a *pressure* P that depend on spectrum
blackbody radiation: $B_\nu(T)$ depends only only T

- so we should have $u(T)$ and $P(T)$
- and blackbody radiation is isotropic, so $P = u/3$

In Director's Cut Extras today:
using $P = u/3$ equation of state in thermodynamic laws gives

$$u(T) \propto T^4 \tag{3}$$

This is Huge! *Q: why?*

can find radiation energy density
just from thermodynamic considerations!

- $u(T) \propto T^4$: strong T dependence!

- can write

$$u(T) = a\, T^4 \tag{4}$$

  where a is the "radiation constant"
  value not determined by thermodynamics alone

- also get total intensity and flux!

$$B(T) = \frac{ac}{4\pi}\, T^4 \tag{5}$$

$$F(T) = \pi B(T) = \frac{ac}{4}\, T^4 \equiv \sigma T^4 \tag{6}$$

same T^4 scaling for total intensity and flux

The Limits of Thermodynamics

We have seen:

blackbody quantities fixed entirely by T

- no adjustable parameters!

- blackbody energy density, pressure independent of V
  unlike nonrelativistic ideal gas! $P_{\text{gas}} = NkT/V = n_{\text{gas}}kT$

- in Extras below: blackbody *entropy density* $s(T) = 4/3\ aT^3$

This is great! But...

- thermodynamics alone does't give radiation constant a
  nor the spectral shape of B_ν
- to find B_ν need a microscopic picture of photons
  from *statistical mechanics*

Statistical Mechanics in a Nutshell

classically, **phase space** $(\vec{x}, \vec{p})$
completely describes particle state

Q: phase space lifestyle of single classical 1-D free body?
of single 1-D harmonic oscillator?
Q: a swarm of free bodies? oscillators?

but quantum mechanics $\rightarrow$ uncertainty $\Delta x \Delta p \geq \hbar/2$

semi-classically:
can show that a quantum particle must occupy
a *minimum* phase space "volume"
$(dx\ dp_x)(dy\ dp_y)(dz\ dp_z) = h^3 = (2\pi\hbar)^3$
per quantum state of fixed $\vec{p}$

Distribution Function

define "**occupation number**" or "**distribution function**" $f(\vec{x}, \vec{p})$:
number of particles in each phase space "cell"

Q: f range for fermions? bosons?

Q: what is f for one classical particle? many classical particles?

Given distribution function, total number of particles is

$$dN = g f(\vec{x}, \vec{p}) \, \frac{d^3\vec{x} \; d^3\vec{p}}{h^3} \tag{7}$$

where g is # internal states: spin/helicity, excitation

Q: $g(e^-)$? $g(\gamma)$? $g(p)$?

particle phase space occupation f determines bulk properties

Q: how? Hint–what's # particles per unit spatial volume?

Fermions: $0 \leq f \leq 1$ (Pauli)

Bosons: $f \geq 0$

internal degrees of freedom

electrons: spin-1/2 gives $g(e^-) = 2s(e^-) + 1 = 2$

protons: also spin-1/2, $g(p) = 2$

photons: $g(\gamma) = 2$ (polarizations)

Particle phase space occupation f determines bulk properties

number density

$$n(\vec{x}) = \frac{d^3N}{d^3x} = \frac{g}{h^3} \int d^3\vec{p}\ f(\vec{p}, \vec{x}) \tag{8}$$

Q: this expressions is general–specialize to photons?

for photons $E = cp = h\nu$
so $d^3p = p^2\,dp\,d\Omega = h^3/c^3\ \nu^2\ d\nu\ d\Omega$

photon number density is thus

$$dn = \frac{2}{c^3}\nu^2\ f(\nu)\ d\nu\ d\Omega \tag{9}$$

and thus we have

$$\frac{dn_\nu}{d\Omega} = \frac{dn}{d\nu\ d\Omega} = \frac{2}{c^3}\nu^2\ f(\nu) \tag{10}$$

thus f *gives a general, fundamental description of photon fields*
the challenge is to find the physics that determines f
$\rightarrow$ spoiler alert: you have already seen a version of it!
but will see it again as the Boltzmann equation!

Note: distribution function $f(\nu)$ and specific intensity I_ν
are *equivalent* and *interchangeable descriptions*
Q: why? how do we get I_ν from $f(\nu)$?

Distribution Function and Observables

distribution function $f(\nu)$ is related to photon number via

$$\frac{dn_\nu}{d\Omega} = \frac{dN}{dV\ d\nu\ d\Omega} = \frac{2}{c^3}\nu^2\ f(\nu) \qquad (11)$$

but we found that photon specific intensity is related to specific number density via

$$\begin{aligned} I_\nu &= \text{energy flux per solid angle} \\ &= \text{energy} \times \text{speed} \times \text{number density per solid angle} \\ &= h\nu\ c\ \frac{dn_\nu}{d\Omega} \qquad (12) \end{aligned}$$

but this means that the two are related via

$$\boxed{I_\nu = \frac{2h}{c^2}\nu^3\ f(\nu)} \qquad (13)$$

so I_ν really is a measure of distribution function f!

Equilibrium Occupation Numbers

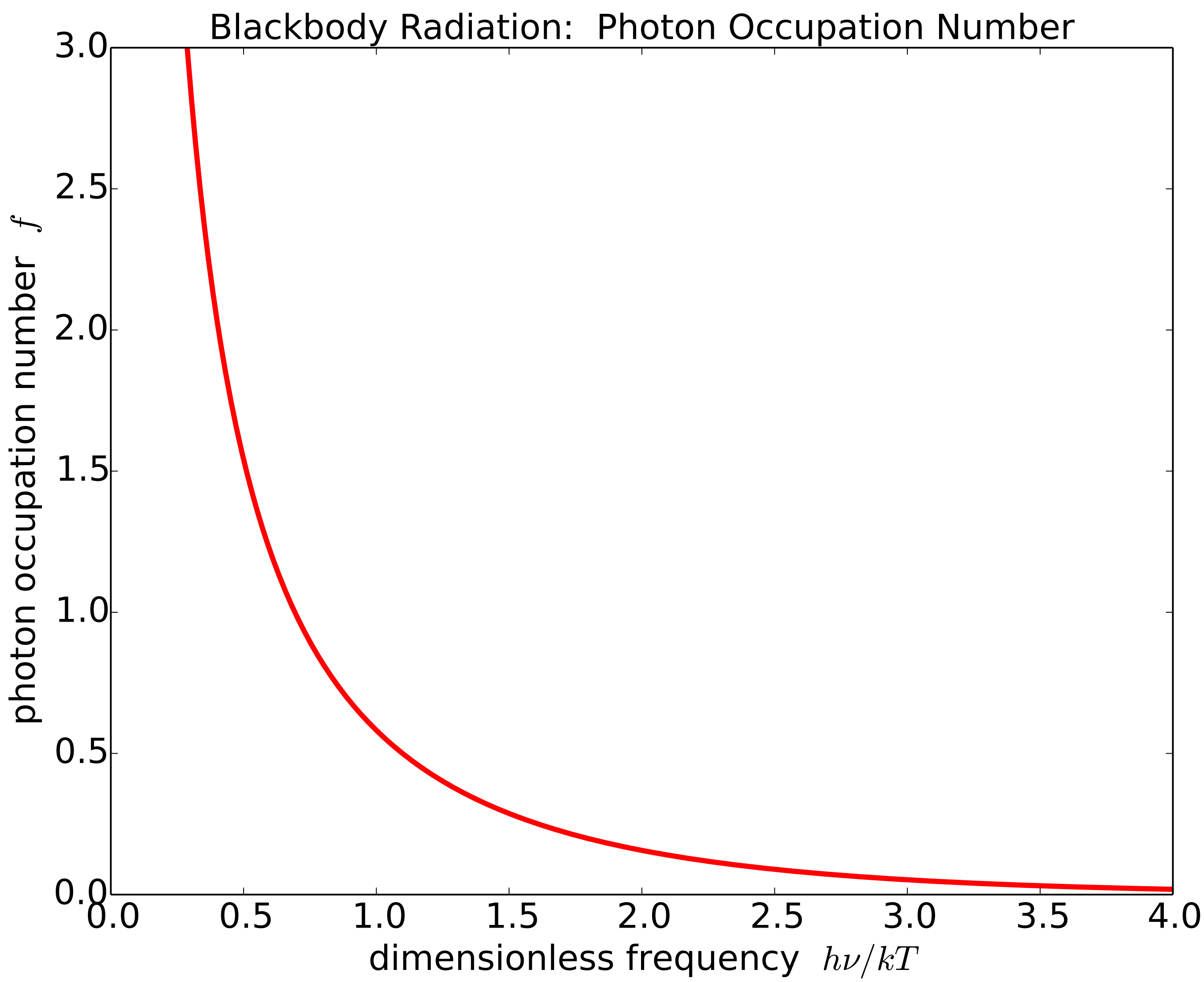
So far, totally general description of photon fields
no assumption of thermodynamic equilibrium

in *thermodynamic equilibrium* at T, the distribution function
is also the *occupation number*
i.e., average *number* of photons with frequency ν

$$f(\nu, T) = \frac{1}{e^{h\nu/kT} - 1} \tag{14}$$

see derivation in today's Director's Cut Extras

Q: at fixed T, for which ν is f large? small?
Q: sketch of $f(\nu)$?
Q: what does this all mean physically?
Q: when is f zero?
Q: in which regime do we expect classical behavior? quantum?


Blackbody Radiation: Photon Occupation Number
photon occupation number f
dimensionless frequency $h\nu/kT$
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0

Blackbody Radiation Properties

Using the blackbody distribution function, we define

$$B_\nu(T) \equiv I_\nu(T) = \frac{2h}{c^2}\nu^3 \; f(\nu, T) \quad (15)$$

and because $f(\nu, T) = 1/(e^{h\nu/kT} - 1)$, we have

$$B_\nu(T) = \frac{2h}{c^2}\frac{\nu^3}{e^{h\nu/kT} - 1} \quad (16)$$

with $h =$ Planck's constant, $k =$ Boltzmann's constant

in wavelength space

$$B_\lambda(T) = 2hc^2\frac{\lambda^{-5}}{e^{hc/\lambda kT} - 1} \quad (17)$$

blackbody integrated intensity:

$$B(T) = \int B_\nu(T)\ d\nu = \int B_\lambda(T)\ d\lambda \tag{18}$$

$$= \frac{2\pi^4}{15}\frac{k^4T^4}{c^3h^3} = \frac{\sigma}{\pi}T^4 = \frac{c}{4\pi}aT^4 \tag{19}$$

blackbody specific and total/integrated flux

$$F_\nu(T) = \pi B_\nu(T) = \frac{2\pi h}{c^2}\frac{\nu^3}{e^{h\nu/kT}-1} \tag{20}$$

$$F(T) = \pi B(T) \equiv \sigma T^4 = \frac{2\pi^5}{15}\frac{k^4T^4}{c^2h^3} \tag{21}$$

defines *Stefan-Boltzmann constant*

$$\sigma = \frac{2\pi^5}{15}\frac{k^4}{c^2h^3} = 5.670 \times 10^{-5}\ \mathrm{erg\ cm^{-2}\ s^{-1}\ K^{-4}} \tag{22}$$

spectral and total/integrated energy density

$$u_\nu(T) = \frac{4\pi B_\nu(T)}{c} = \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/kT} - 1} \tag{23}$$

$$u(T) = \frac{4\pi B(T)}{c} = \frac{8\pi^5}{15} \frac{k^4}{c^3 h^3} T^4 \tag{24}$$

$$\equiv aT^4 = \frac{4\sigma}{c} T^4 \tag{25}$$

and now we find the value of a!

Stefan-Boltzmann radiation density constant

$$a = \frac{4\sigma}{c} = 7.56 \times 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4} \tag{26}$$

at last!

Q: to order of magnitude: integrated number density?

mean number density: dimensions $[n] = [\text{length}^{-3}]$
can only depend on T, and physical constants h, c, k
can form only one length: $[hc/kT] = [\text{length}]$
$\rightarrow$ expect $n \sim (hc/kT)^3$

photon number density

$$n_\nu(T) = \frac{4\pi B_\nu(T)}{hc\nu} = \frac{8\pi}{c^3} \frac{\nu^2}{e^{h\nu/kT} - 1} \quad (27)$$

$$n(T) = 16\pi\zeta(3) \left(\frac{kT}{hc}\right)^3 \quad (28)$$

where $\zeta(3) = 1 + 1/2^3 + 1/3^3 + 1/4^3 + \cdots = 1.2020569\ldots$

Q: implications?

blackbody photon number density

$$n(T) = 16\pi\zeta(3) \left(\frac{kT}{hc}\right)^3 \tag{29}$$

i.e., $n \propto T^3$

So if temperatures changes, photon number changes

blackbody photon number is not conserved

photons massless $\rightarrow$ can always make more!

if heat up, photon number increases

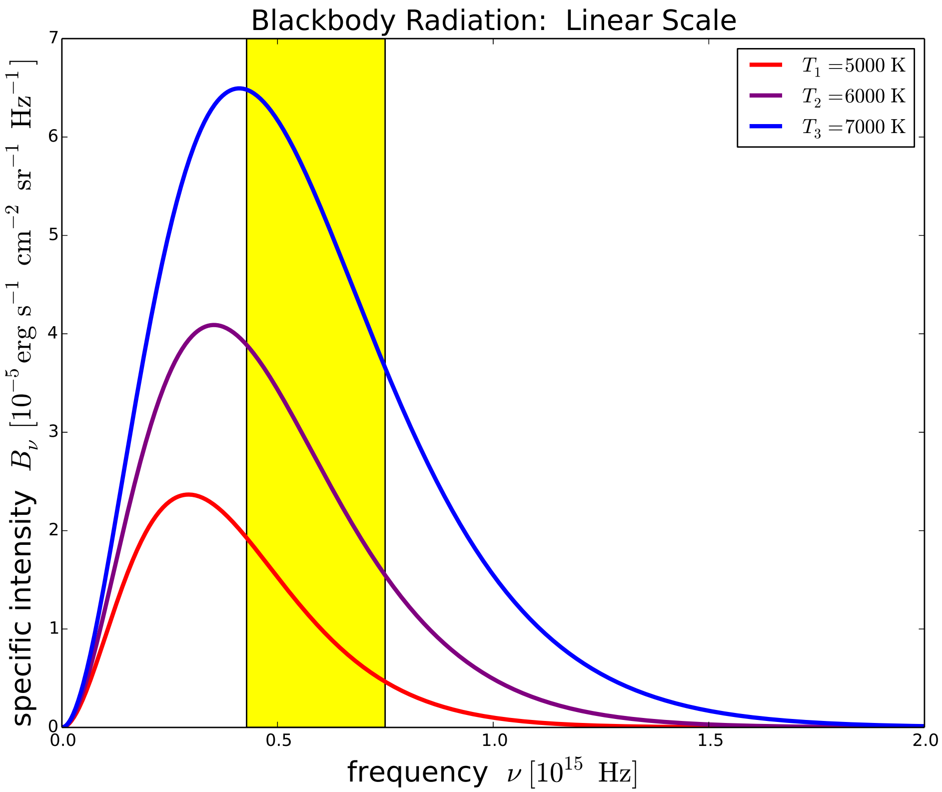
and spectrum relaxes to blackbody form

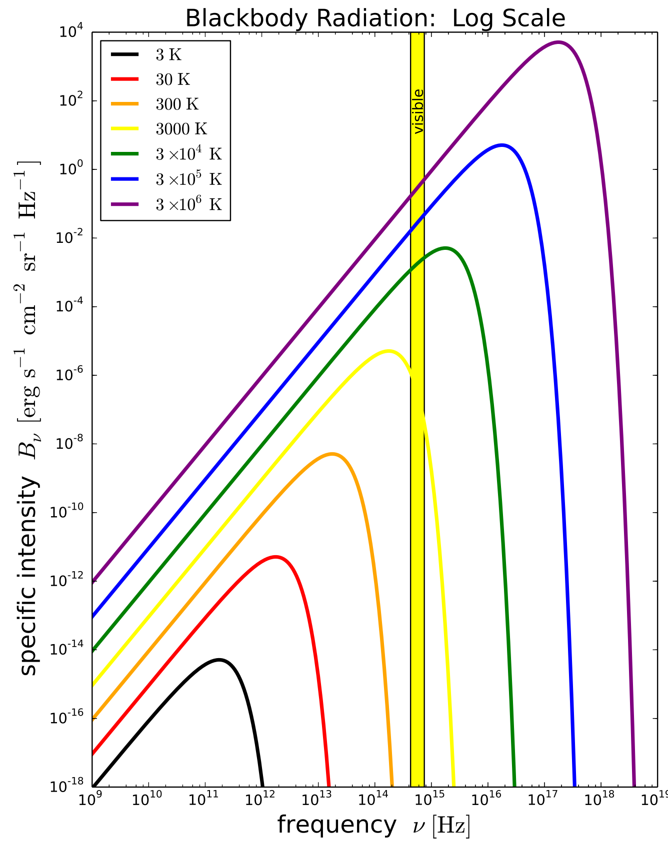
alternatively: given energy density $u \sim T^4$

and mean photon energy $\langle E \rangle \sim kT$

number density must be $n \sim T^3$

Blackbody Spectral Properties





plots of B_ν vs ν *Q: what strikes you?*

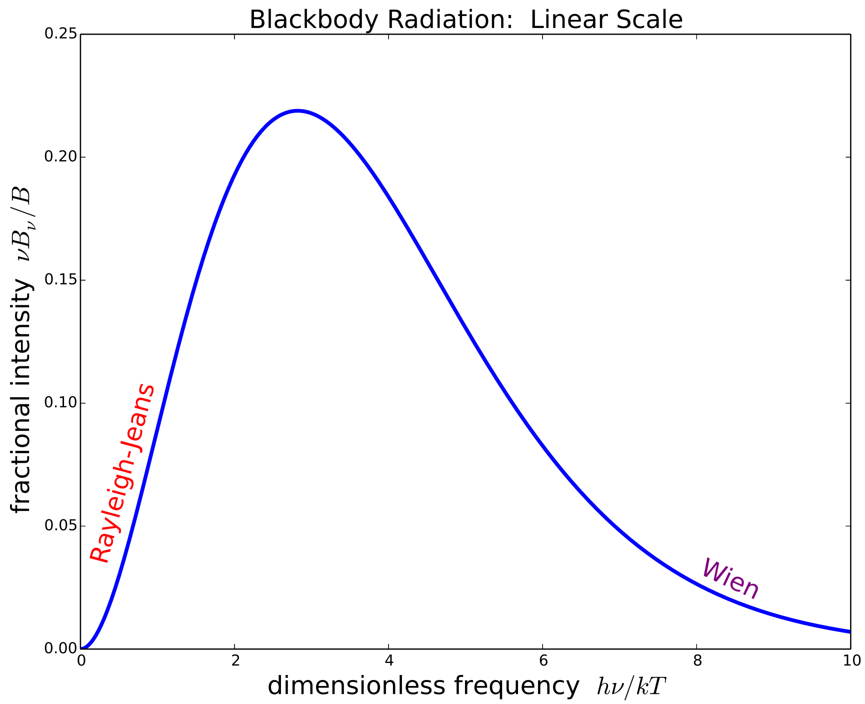
Blackbody Spectral Properties

at fixed ν, occupation number $\partial_T f(\nu, T) > 0$
$\rightarrow$ at larger T: larger f and so more photons
$\rightarrow$ also more specific intensity, flux, energy density
$\rightarrow$ slogan: *"blackbody spectra at different T never cross"*

natural energy scale kT, sets two limits

Rayleigh-Jeans limit $h\nu \ll kT$
occupation number $f(\nu) \rightarrow kT/h\nu \gg 1$
  many photons, expect classical behavior
specific intensity $I_\nu = 2h/c^2\ \nu^3 f \rightarrow 2kT\,\nu^2/c^2$

- $I_\nu \propto \nu^2$: power-law scaling
- h does not appear in I_ν: classical behavior!



Wien limit $h\nu \gg kT$

occupation number $f(\nu) \to e^{-h\nu/kT} \ll 1$

photon starved: thermal bath cannot "pay energy cost"

specific intensity $I_\nu \to 2h\,\nu^3/c^2 \; e^{-h\nu/kT}$

- exponentially damped due to quantum effects

Director's Cut Extras

Thermodynamics Recap

First Law of Thermodynamics: heat is work!
adding *heat energy* dQ to system changes
system *energy* U and/or *pressure* P:

$$dQ = dU + pdV \tag{30}$$

Second Law of Thermodynamics: heat is entropy!

$$T\ dS = dQ \tag{31}$$

together

$$T\ dS = dU + P\,dV \tag{32}$$

and thus entropy $S = S(T, V)$ obeys

$$dS = \frac{dU}{T} + \frac{P}{T}dV \tag{33}$$

entropy $S = S(T, V)$ obeys

$$dS = \frac{dU}{T} + \frac{P}{T} dV \tag{34}$$

and thus we have

$$\partial_T S = \frac{\partial_T U}{T} \tag{35}$$

$$\partial_V S = \frac{\partial_V U + P}{T} \tag{36}$$

which means

$$\partial_V \partial_T S = \frac{\partial_V \partial_T U}{T} \tag{37}$$

$$\partial_T \partial_V S = \frac{\partial_T \partial_V U}{T} - \frac{\partial_V U}{T^2} + \partial_T \left(\frac{P}{T}\right) \tag{38}$$

but mix partial derivatives equal, e.g., $\partial_V \partial_T S = \partial_T \partial_V S$, and note that $\partial_V U|_T = u$ energy density, so

$$u = T^2 \; \partial_T \left(\frac{P}{T}\right) \tag{39}$$

Radiation Thermodynamics

general thermodynamic considerations give:

$$u = T^2 \ \partial_T \left(\frac{P}{T} \right) \tag{40}$$

now specialize to *radiation:* $P = P(T) = u(T)/3$

$$T \frac{d}{dT} \left(\frac{u}{T} \right) = 3 \frac{u}{T} \tag{41}$$

which gives

$$\frac{d(u/T)}{u/T} = 3 \ \frac{dT}{T} \tag{42}$$

$$\ln \left(\frac{u}{T} \right) = 3 \ln(T) + \ln(a) \tag{43}$$

$$u(T) = a \ T^4 \tag{44}$$

This is Huge! *Q: why?*

Radiation Entropy

Using $U = aT^4V$ and $P = u/3$, can solve for
radiation entropy

$$S_{\rm rad} = \frac{4}{3} a T^3 \; V \tag{45}$$

and thus *entropy density* $s_{\rm rad}(T) = S/V = 4/3 \; aT^3$

if entropy $S_{\rm rad}$ constant in a parcel of radiation
$\rightarrow$ *adiabatic* process:

$$T_{\rm adiabat} \propto V^{-1/3} \tag{46}$$

$$P_{\rm adiabat} \propto T^4_{\rm adiabat} \propto V^{-4/3} \tag{47}$$

writing $P \propto V^{-\gamma}$, we have
an *adiabatic index* $\gamma_{\rm rad} = 4/3$

Q: but how do we get the radiation constant a?

Blackbody Photon Occupation Number

at a fixed temperature T and frequency ν
we want the distribution function f, i.e., the occupation number
i.e., the average number of photons with frequency ν

Boltzmann: probability of having state n of energy E_n
proportional to $p_n = e^{-E_n/kT}$

Planck: n photons have $E_n = n\,h\nu$, so $p_n = e^{-nx}$
with $x = h\nu/kT$

So average number is

$$f = \langle n \rangle = \frac{\sum_n n p_n}{\sum_n p_n} = \frac{\sum_n n e^{-nx}}{\sum_n e^{-nx}} \tag{48}$$

note that $\sum_n n e^{-nx} = -\partial_x \sum_n e^{-nx}$, so

$$f = -\partial_x \ln\left(\sum_n e^{-nx}\right) \tag{49}$$

but geometric series has sum

$$\sum_n e^{-nx} = \sum_n (e^{-x})^n = \frac{1}{1 - e^{-x}} \tag{50}$$

and thus

$$\begin{aligned} f &= -\partial_x \ln \frac{1}{1 - e^{-x}} = \partial_x \ln(1 - e^{-x}) & (51) \\ &= \frac{e^{-x}}{1 - e^{-x}} & (52) \end{aligned}$$

which gives

$$f(\nu, T) = \frac{1}{e^{h\nu/kT} - 1} \tag{53}$$

which was to be shewn

Average Energy per Blackbody Photon

only one way to form an energy
$\rightarrow$ expect $\langle E \rangle \sim kT$; exact result:

$$\langle E \rangle \equiv \frac{u(T)}{n(T)} \quad (54)$$

$$= \frac{\pi^4}{30\zeta(3)} kT = 2.701\, kT \quad (55)$$

c.f. nonrelativistic ideal gas: $\langle E \rangle_{\rm idealgas} = 3/2kT$

note: blackbody radiation has

$$\frac{P}{n\ kT} = \frac{\langle E \rangle}{3} = 0.900 \quad (56)$$

c.f. nonrelativistic ideal gas: $P_{\rm idealgas}/n_{\rm idealgas} kT = 1$

Average Entropy per Blackbody Photon

mean entropy per photon:
entropy has units of Boltzmann's k
$\rightarrow$ expect $\langle S \rangle \sim k$; exact result

$$\langle S \rangle = \frac{s(T)}{n(T)} = \frac{4u(T)/3T}{n(T)} = \frac{4}{3}\frac{\langle E \rangle}{T} = 3.601\, k \tag{57}$$

temperature independent!

c.f. nonrelativistic ideal gas: entropy per particle
given by Sackur-Tetrode equation

$$\frac{s_{\rm idealgas}}{n_{\rm idealgas}} = k \left[\frac{5}{2} - \ln \left(\frac{n}{(2\pi mkT/h)^{3/2}} \right) \right] \tag{58}$$

nearly constant, but through logarithm term
weakly depends on T and density n