Astronomy 501: Radiative Processes

Lecture 9

Sept 12, 2022

Announcements:

- **Problem Set 3 due Friday**

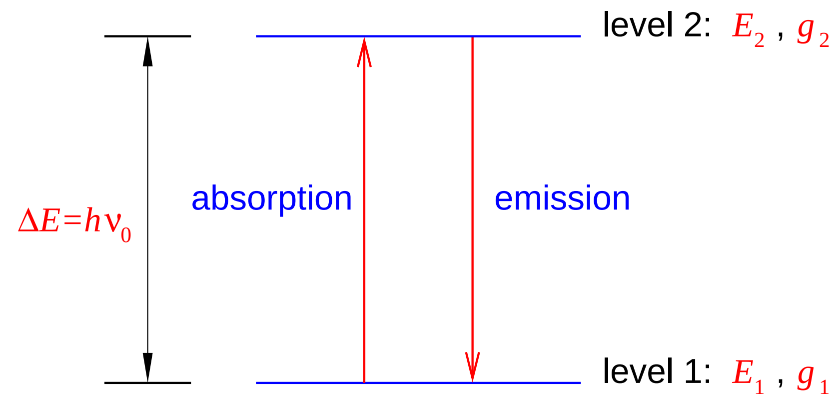
Last time: radiative properties of a two-level system

Q: emission processes? absorption? what do they depend on?

2-level system

two states: energies E_1, E_2

degeneracies g_1, g_2



absorption depends on local radiation field

$$\text{transition probability per time per atom} \quad = B_{12}\bar{J} \qquad (1)$$

spontaneous emission independent of local conditions

$$\text{transition probability per unit time per atom} = A_{21} \qquad (2)$$

stimulated emission “peer pressure” by local radiation field

$$\text{transition probability per time per atom} \quad = B_{21}\bar{J} \qquad (3)$$

Two-Level Systems: Thermal Radiation

Now consider the two-level atom as a radiating system
What are the emission and absorption coefficients?

Emission Coefficient

spontaneous emission rate per atom in state 2: A_{21}
$\rightarrow$ rate per volume: $n_2 A_{21}$
$\rightarrow$ total power emitted per volume: $h\nu_0 \, n_2 A_{21}$
emission isotropic $\rightarrow$ power per volume per solid angle:
$h\nu_0 \, n_2 A_{21} / 4\pi$ *Q: why?*

but still need *frequency spectrum*
of emitted radiation, i.e., *emission profile*

Q: simplest assumption?

simplest assumption (generally accurate):
$\rightarrow$ emission spectrum profile = absorption profile $\phi(\nu)$

and thus energy released in spontaneous emission is

$$d\mathcal{E} = \frac{h\nu_0}{4\pi} n_2 A_{21} \phi(\nu) \; dV \; dt \; d\nu \; d\Omega \qquad (4)$$

and thus the emission coefficient is

$$j_\nu = \frac{h\nu_0}{4\pi} n_2 A_{21} \phi(\nu) \qquad (5)$$

absorption coefficient

absorption rate per atom in level 1: $B_{12}\bar{J}$

thus energy absorbed is

$$d\mathcal{E} = \frac{h\nu_0}{4\pi} n_1 B_{12} \bar{J} \; dV \; dt \tag{6}$$

but $4\pi \bar{J} = \int d\Omega \int I_\nu \phi(\nu) \; d\nu$, so

$$d\mathcal{E} = \frac{h\nu_0}{4\pi} n_1 B_{12} I_\nu \phi(\nu) \; dV \; dt \; d\Omega \; d\nu \tag{7}$$

recall: path element ds in area dA has volume $dV = ds \; dA$ and so we find absorption coefficient

$$\alpha_{\mathrm{abs},\nu} = \frac{h\nu_0}{4\pi} n_1 B_{12} \phi(\nu) \tag{8}$$

...but we are not done! *Q: because...?*

stimulated emission

tempting to include this as additional emission term

but wait! stimulated emission depends on (average) intensity
$\rightarrow$ formally more similar to absorption

formally better to treat stimulated emission as
a *negative absorption* term:

$$\alpha_{\mathrm{stim},\nu} = -\frac{h\nu_0}{4\pi} n_2 B_{21} \phi(\nu) \tag{9}$$

and then *(net) absorption coefficient*

$$\alpha_\nu = \alpha_{\mathrm{abs},\nu} + \alpha_{\mathrm{stim},\nu} \tag{10}$$

$$= \frac{h\nu_0}{4\pi} \phi(\nu)\ (n_1 B_{12} - n_2 B_{21}) \tag{11}$$

Two-Level Radiation Transfer

Transfer equation for two-level atom

$$\frac{dI_\nu}{ds} = -\frac{h\nu_0}{4\pi}\phi(\nu)\ \left(n_1 B_{12} - n_2 B_{21}\right) I_\nu + \frac{h\nu_0}{4\pi} n_2 A_{21} \phi(\nu) \tag{12}$$

source function

$$S_\nu = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}} \tag{13}$$

Einstein relations give

$$\alpha_\nu = \frac{h\nu_0}{4\pi}\phi(\nu)\ n_1 B_{12} \left(1 - \frac{n_2}{n_1}\frac{g_1}{g_2}\right) \tag{14}$$

$$S_\nu = \frac{2h\nu^3/c^2}{(n_1/n_2)(g_2/g_1) - 1} \tag{15}$$

a generalization of Kirchhoff's laws

these *do not assume thermal equilibrium!*

Q: interesting cases?

Local Thermodynamic Equilibrium

if atom levels are in thermodynamic equilibrium
then we have $n_1/n_2 = (g_1/g_2)e^{h\nu/kT}$ and

$$S_\nu = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1} = B_\nu(T) \tag{16}$$

we recover the usual Kirchhoff's law! as we must!

and absorption term becomes

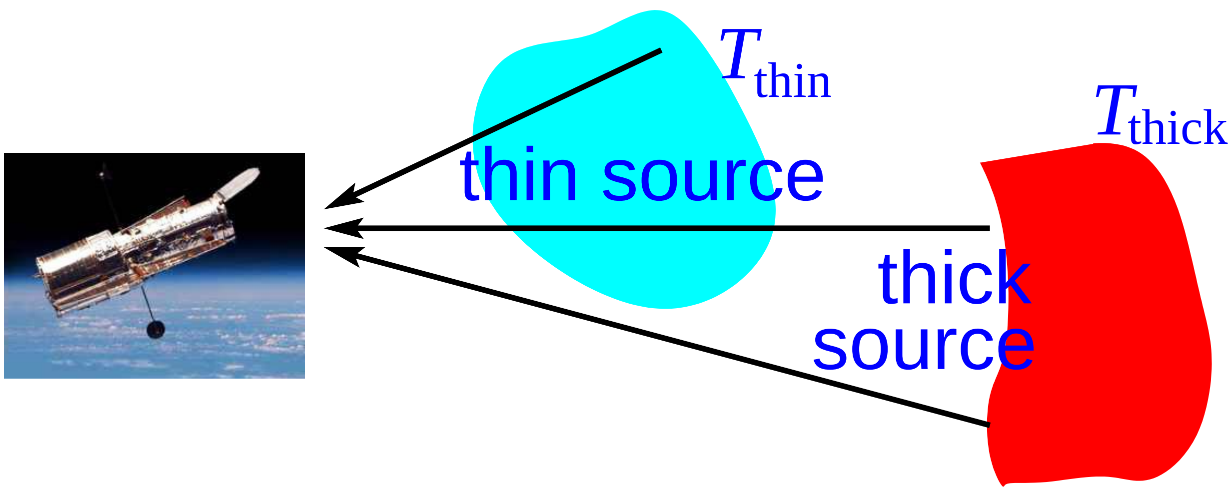
$$\alpha_\nu = \frac{h\nu_0}{4\pi}\phi(\nu)\ n_1 B_{12}\left(1 - e^{-h\nu/kT}\right) \tag{17}$$

i.e., "uncorrected" term minus stimulated emission correction

What if not in thermodynamic equilibrium?
then $n_2/n_1 \neq$ Boltzmann expression
emission is nonthermal

Kirchhoff's Laws in Action

Consider matter including 2-level system
$\Delta E = h\nu_0$, in equilibrium at T



Q: What spectrum if optically thin (and not backlit)?

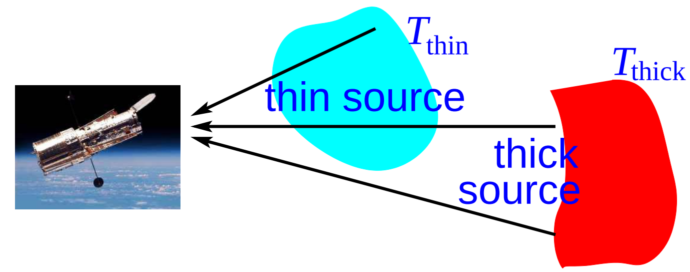
Q: What spectrum if optically thick (and no foreground matter)?

Q: astro examples?

Kirchhoff: Single Source

Kirchhoff: thermal source has $j_\nu(T) = \alpha_\nu(T)\ B_\nu(T)$
2-level: $j_\nu \propto \alpha_\nu \propto \tau_\nu \propto \phi(\nu)$ line profile

single source at T

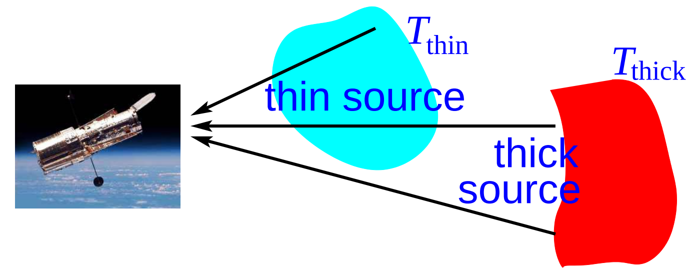


- optically thin: $I_\nu = j_\nu(T)\ \delta s$
  see *line at ν_0 in emission*
- optically thick: $I_\nu = S_\nu = B_\nu$ *blackbody, Planck function*

Now consider same matter, but
- optically thick source at $T_{\rm thick}$
- enshrouded by optically thin source at $T_{\rm thin}$

Q: spectrum seen only through thin source?
Q: spectrum thru thick source if $T_{\rm thick} > T_{\rm thin}$? if $T_{\rm thick} > T$thin?
Q: astro examples of each case?

two sources at T_{thick} and T_{thin}



- thin source only: emission line spectrum
- for thin source backlit by thick:

$$I_\nu \approx (1 - \tau_\nu)\, B_\nu(T_{\text{thick}}) + \tau_\nu B_\nu(T_{\text{thin}}) \quad (18)$$

$$= B_\nu(T_{\text{thick}}) + \tau_\nu \left[B_\nu(T_{\text{thin}}) - B_\nu(T_{\text{thick}})\right] \quad (19)$$

- $\nu \neq \nu_0$: $I_\nu \approx B_\nu(T_{\text{thick}})$
  observe *Planck continuum* of thick source

- around ν_0: $T_{\text{thick}} > T_{\text{thin}}$ gives *absorption line* in continuum
  $T_{\text{thick}} < T_{\text{thin}}$ gives *emission line* above continuum

`www: examples`

Inverted Populations

two-level absorption coefficient is:

$$\alpha_\nu = \frac{h\nu_0}{4\pi}\phi(\nu)\ n_1 B_{12}\left(1 - \frac{n_2}{n_1}\frac{g_1}{g_2}\right) \tag{20}$$

note that the algebraic *sign*

depends on population levels, i.e., on n_2/n_1

normally, lower level more populated: $n_1 > n_2$

If we can arrange or stumble upon a system where

$$\frac{n_1}{g_1} < \frac{n_2}{g_2} \tag{21}$$

i.e., an *inverted population*, then $\alpha_\nu < 0$!

Q: and then what happens to propagating light?

Q: examples?

Q: how might we arrange an inverted population?

Masers

if $\alpha_\nu < 0$, then propagating light has
exponential *increase* in intensity!

stimulated emission causes a "cascade" of photons

in lab: create inverted populations of atoms
use mirrors to "recycle" stimulating photons
$\rightarrow$ *this is a laser!* light amplification by stimulated emission of radiation

in cosmos: inverted populations of molecules
maser: microwave amplification by stimulated emission of radiation

how to create inversion?
need *nonthermal* mechanism to "pump" upper level

Scattering

Pure Scattering

Consider an idealized case with radiation propagating through a medium with "*pure scattering*," i.e., scattering, but *no emission*, and *no absorption*

Recall: intensity in a ray is a directional quantity
i.e., really $I_\nu = I_\nu(\theta, \phi) = I_\nu(\hat{n})$,
with $\hat{n}$ a unit vector toward (θ, ϕ)

in general: scattering preserves photon *number*
but *redistributes* both

- photon energy
- photon direction

generally, scattering is different for different incident and scattered angles, i.e., anisotropic

this is generally is (very) non-trivial to calculate but consider even more special case:

- *isotropic* scattering:
  no preferred direction for scattered photons
- photon energy unchanged ("*coherent scattering*")
  good approximation for scattering by non-relativistic e

define **scattering coefficient** $\varsigma_\nu = n_{\rm scat}\sigma_{{\rm scat},\nu}$,
and thus also scattering cross section $\sigma_{\rm scat}$, such that
intensity lost to scattering *out* of ray is

$$dI_\nu = -\varsigma_\nu \; I_\nu \; ds \qquad (22)$$

isotropic scattering $\rightarrow \varsigma_\nu$ same for all directions

Q: what is intensity scattered into *the ray?*

Isotropic Coherent Scattering

intensity scattered *out* of ray $I_\nu(\hat{n})$ is

$$dI_\nu(\hat{n}) = -\varsigma_\nu \; I_\nu(\hat{n}) \; ds \tag{23}$$

if scattering *isotropic*, the portion *into* $\hat{n}$ is

$$dI_\nu(\hat{n}) = \frac{d\Omega'}{4\pi} \left| dI_\nu(\hat{n}') \right| \tag{24}$$

and so integrating over all possible solid $d\Omega'$ gives

$$dI_\nu(\hat{n}) = \frac{\varsigma_\nu}{4\pi} \int I_\nu \; d\Omega \; ds = \varsigma_\nu \; J_\nu \; ds \tag{25}$$

where J_ν is the angle-averaged intensity and thus for isotropic coherent scattering

$$\frac{dI_\nu(\hat{n})}{ds} = -\varsigma_\nu \left[I_\nu(\hat{n}) - J_\nu \right] \tag{26}$$

and so the source function is

$$S_\nu = J_\nu \tag{27}$$

and the transfer equation can be written

$$\frac{dI_\nu(\hat{n})}{d\tau_\nu} = -I_\nu(\hat{n}) + J_\nu \tag{28}$$

where mean flux $J_\nu = \int I_\nu(\hat{n}')d\Omega'/4\pi$, and $d\tau_\nu = \varsigma_\nu \, ds$

Q: why is this intuitively correct?

Q: what is effect on I_ν of many scattering events?

for coherent, isotropic scattering:

$$\frac{dI_\nu(\hat{n})}{d\tau_\nu} = -I_\nu(\hat{n}) + J_\nu \tag{29}$$

depends on I_ν field in *all directions*

$\Rightarrow$ scattering couples intensity in different directions

if many scattering events, τ_ν large: $I_\nu \to J_\nu$

after large number of mean free paths, photons $\to$ isotropic

$\Rightarrow$ (isotropic) scattering randomizes photon directions
reduces anisotropy

transfer with scattering: integro-differential equation
generally very hard to solve!

Q: transfer equation modification for anisotropic scattering?

Scattering and Random Walks

Can we understand photon propagation with isotropic scattering in a simple physical picture?

simple model: **random walk**
between collisions, photons move in straight "steps"
with random displacement $\hat{\ell}$
position after N collisions ("steps") is $\vec{r}_N$

idealizations:

- *step length uniform*: $|\hat{\ell}| = \ell_{\rm mfp}$ mean free path
- *step direction random*: each $\hat{\ell}$ drawn from isotropic distribution and independent of previous steps
- initial condition: start at center, $\vec{r}_0 = 0$

1. first step $\vec{r}_1 = \hat{\ell}$

length $|\vec{r}_1| = \ell_{\rm mfp}$, direction random

average over ensemble of photons:

- $\langle \vec{r}_1 \rangle = 0$
- $\left\langle r_1^2 \right\rangle = \ell_{\rm mfp}^2$

average positions for *ensemble* of photons is zero

but average distance of *each* photon $\ell_{\rm mfp}$

2. step N has: $\vec{r}_N = \vec{r}_{N-1} + \hat{\ell}$

average over ensemble of photons:

$\langle \vec{r}_N \rangle = \langle \vec{r}_{N-1} \rangle + \left\langle \hat{\ell} \right\rangle = \langle \vec{r}_{N-1} \rangle$

but by recursion

$$\langle \vec{r}_N \rangle = \langle \vec{r}_{N-1} \rangle = \langle \vec{r}_{N-2} \rangle = \ldots = \langle \vec{r}_1 \rangle = 0 \tag{30}$$

$\rightarrow$ ensemble average of photons displacements still 0

as it must be by symmetry

but what about *mean square* displacement?

$$r_N^2 = \vec{r}_N \cdot \vec{r}_N \tag{31}$$

$$= r_{N-1}^2 + 2\hat{\ell} \cdot \vec{r}_{N-1} + \ell_{\rm mfp}^2 \tag{32}$$

average over photon ensemble

$$\left\langle r_N^2 \right\rangle = \left\langle r_{N-1}^2 \right\rangle + 2\left\langle \hat{\ell} \cdot \vec{r}_{N-1} \right\rangle + \ell_{\rm mfp}^2 \tag{33}$$

Q: what is $\left\langle \hat{\ell} \cdot \vec{r}_{N-1} \right\rangle$*?*

$$\left\langle r_N^2 \right\rangle = \left\langle r_{N-1}^2 \right\rangle + 2 \left\langle \hat{\ell} \cdot \vec{r}_{N-1} \right\rangle + \ell_{\rm mfp}^2 \quad (34)$$

each photon scattering direction independent from previous

$\left\langle \hat{\ell} \cdot \vec{r}_{N-1} \right\rangle = \ell_{\rm mfp} r_{N-1} \langle \cos\theta \rangle = 0$

so $\left\langle r_N^2 \right\rangle = \left\langle r_{N-1}^2 \right\rangle + \ell_{\rm mfp}^2$

but this means $\left\langle r_N^2 \right\rangle = N \ell_{\rm mfp}^2$

$\rightarrow$ **each** photon goes r.m.s. distance

$$r_{\rm rms} = \sqrt{\left\langle r_N^2 \right\rangle} = \sqrt{N}\ \ell_{\rm mfp} \quad (35)$$

so imagine photons generated at $r = 0$

and, after scattering, are observed at distance L

Q: number N of scatterings if optically thin? thick?

Photon Random Walks and Optical Depth

if travel distance L by random walk
then after N scatterings $L = \sqrt{N}\ \ell_{\rm mfp}$
but photon optical depth is $\tau = L/\ell_{\rm mfp}$
$\rightarrow$ counts number of mean free paths in length L

optically thick: $\tau \gg 1$
many scattering events $\rightarrow$ *this is a random walk!*
$N \stackrel{\rm thick}{\approx} \tau^2$

if *optically thin*: $\tau \ll 1$
scattering probability $1 - e^{-\tau} \approx \tau \ll 1$: *not random walk!*
mean number of scatterings over L is $N \stackrel{\rm thin}{\approx} \tau$

approximate expression good for all τ

$$N \approx \tau + \tau^2 \tag{36}$$

Combined Scattering and Absorption

generally, matter can both scatter and absorb photons
transfer equation must include both
for *coherent isotropic scattering* of *thermal radiation*

$$\frac{dI_\nu}{ds} = -\alpha_\nu (I_\nu - B_\nu) - \varsigma_\nu (I_\nu - J_\nu) \tag{37}$$

giving a source function

$$S_\nu = \frac{\alpha_\nu B_\nu + \varsigma_\nu J_\nu}{\alpha_\nu + \varsigma_\nu} \tag{38}$$

a *weighted average* of the two source functions

thus we can write

$$\frac{dI_\nu}{ds} = -(\alpha_\nu + \varsigma_\nu)(I_\nu - S_\nu) \tag{39}$$

with **extinction coefficient** $\alpha_\nu + \varsigma_\nu$

generalize mean free path:

$$\ell_{\mathrm{mfp},\nu} = \frac{1}{\alpha_\nu + \varsigma_\nu} \tag{40}$$

average distance between photon interactions

in random walk picture:
probability of step ending in *absorption*

$$\epsilon_\nu \equiv \alpha_\nu \ell_{\mathrm{mfp},\nu} = \frac{\alpha_\nu}{\alpha_\nu + \varsigma_\nu} \tag{41}$$

and thus step *scattering probability*

$$\varsigma_\nu \ell_{\mathrm{mfp},\nu} = \frac{\varsigma_\nu}{\alpha_\nu + \varsigma_\nu} = 1 - \epsilon_\nu \tag{42}$$

also known as **single scattering albedo**

source function:

$$S_\nu = \epsilon_\nu B_\nu + (1 - \epsilon_\nu) J_\nu \tag{43}$$

Random Walk with Scattering and Absorption

in *infinite medium*: every photon created is eventually absorbed
typical absorption path $\ell_{\mathrm{abs},\nu} = 1/\alpha_\nu$
typical number of scattering events until absorption is

$$N_{\mathrm{scat}} = \frac{\ell_{\mathrm{abs},\nu}}{\ell_{\mathrm{mfp},\nu}} = \frac{\varsigma_\nu + \alpha_\nu}{\alpha_\nu} = \frac{1}{\epsilon_\nu} \quad (44)$$

so typical distance traveled between creation and absorption

$$\ell_* = \sqrt{N_{\mathrm{scat}} \ell_{\mathrm{mfp},\nu}} = \sqrt{\ell_{\mathrm{abs},\nu} \ell_{\mathrm{mfp},\nu}} = \frac{1}{\sqrt{\alpha_\nu(\alpha_\nu + \varsigma_\nu)}} \quad (45)$$

diffusion/thermalization length or *effective mean free path*

What about a *finite medium* of size s?
define optical thicknesses $\tau_{\mathrm{scat}} = \varsigma_\nu s$, $\tau_{\mathrm{abs}} = \alpha_\nu \ s$
and $\tau_* = s/\ell_* = \tau_{\mathrm{scat}}^{1/2}(\tau_{\mathrm{scat}} + \tau_{\mathrm{abs}})^{1/2}$

Q: expected behavior if $\tau_* \ll 1$*?* $\tau_* \gg 1$*?*

$\tau_* = s/\ell_*$: path in units of photon travel
until absorption

$\tau_* \ll 1$: *effectively thin* or *translucent*
photons random walk by scattering, seen before absorption
luminosity of thermal source with volume V is

$$L_\nu \stackrel{\text{thin}}{=} 4\pi\alpha_\nu B_\nu V = 4\pi j_\nu(T) V \tag{46}$$

$\tau_* \gg 1$: *effectively tick*
thermally emitted photons scattered then absorbed before seen
expect $I_\nu \to S_\nu \to B_\nu$
rough estimate of luminosity of thermal source:
most emission from "last scattering" surface of area A
where photons travel $s = \ell_*$

$$L_\nu \stackrel{\text{thick}}{\approx} 4\pi\alpha_\nu B_\nu \ell_* A \approx 4\pi \sqrt{\epsilon_\nu}\, B_\nu A \tag{47}$$